Mathematics of Finance
Problem Set 3
Due October 5

I. Complete Exercises 2.17, 2.24, 3.12, 3.16, 3.19, 4.5, 4.6, 4.7, and 4.8 of Zawstaniak and Capiński.

II. Given a sequence of independent fair coin tosses, write $X_i = 1$ if the ith toss comes up heads and 0 if it comes up tails. Which of the following are martingales? (Justify your answers.)

(a) $S(0) = 0$ and $S(n) = \sum_{i=1}^{n} X_i$ for $n \geq 1$

(b) $S(0) = 1$ and $S(n) = \prod_{i=1}^{n} X_i$ for $n \geq 1$

(c) $S(0) = 1$ and $S(n) = \prod_{i=1}^{n} Y_i$ for $n \geq 1$, where $Y_i = 2X_i$.

(d) $S(0) = 0$ and $S(n) = \sum_{i=1}^{n} Y_i$ for $n \geq 1$, where $Y_i = X_i - \frac{1}{2}$.

(e) $S(0) = 0$ and $S(n) = \sum_{i=1}^{n} i^2 Y_i$ for $n \geq 1$, where $Y_i = 2X_i - 1$

(e) $S(0) = 0$ and $S(n) = \sum_{i=1}^{n} A_i Y_i$ for $n \geq 1$, where $Y_i = 2X_i - 1$ and $A_i = \sum_{j=1}^{i+2} X_j$

(f) $S(0) = 0$ and $S(n) = X_n$ for $n \geq 1$

(g) $S(n) = 7$ for all n

III. Consider the three-step binomial tree model with initial bond price $A(0) = 1$, initial stock price $S(0) = 5$, risk-free interest rate $r = .1$, $u = .3$, $d = -.1$, and $p = 2/3$. That is, $A(n) = (1 + r)^n$ for $n \in \{0, 1, 2, 3\}$; $S(0) = 5$; and the returns $K(n) = \frac{S(n)-S(n-1)}{S(n-1)}$ are independent and identically distributed for $n \in \{1, 2, 3\}$, with each $K(n)$ equal to u with probability p and d with probability $1 - p$.

(a) List the eight possibilities for the sequence of values $S(1), S(2), S(3)$.

(b) Compute the probability of each of these eight sequences.

(c) Compute the risk neutral probability of each of these sequences.

(d) Check that the probabilities in (b) and (c) depend only on the terminal value $S(3)$.

(d) Compute the price of a call option with exercise time 3 and strike price 1.2.

(e) Describe an admissible (i.e., self-financing, predictable, wealth always non-negative with probability one) investment strategy for replicating this option by using combinations of stocks and bonds. (This can be done by drawing the tree and writing at each node in the tree the amount of stock and bond the investor should hold during the unit of time after that node is reached.)

(f) Let $V(n)$ be the wealth of an investor using the strategy described in (e) at time n. Is $V(n)$ a martingale with respect to ordinary probability? With respect to risk neutral probability? Explain your answers.