Mathematics of Finance
Problem Set 1 Solutions

1. (Like Ross, 1.7) Two cards are randomly selected from a deck of 52 playing cards. What is the probability that they are both aces? What is the conditional probability that they are both aces, given that they are of different suits?

ANSWER: (a) Number of two-ace ordered pairs: 4×3. Total number of ordered pairs: 52×51. Probability: $\frac{4\times 3}{52\times 51} = \frac{1}{221}$. (b) $\left(\frac{4}{52}\right)\left(\frac{3}{39}\right) = \frac{1}{169}$.

2. Suppose 10 people toss their hats into a box and then take turns drawing hats at random, so each person ends up with one hat. How many possible permutations of the hats are there? Assume that each such permuation is equally likely. Let S_i be one if the ith person gets her own hat, zero otherwise.

(a) Calculate mean and variance of S_i.

(b) Calculate the covariance of S_i and S_j when $i \neq j$.

(c) Let S be the number of people who get their own hat. Calculate the mean and variance of S.

ANSWER:

(a) $E(S_i)$ is the probability that i gets own hat, which is $\frac{1}{10}$.
$\mathrm{Var}(S_i) = E(S_i^2) - [E(S_i)]^2 = \frac{1}{10} - \frac{1}{100} = \frac{9}{100}$.

(b) Covariance is $E(S_iS_j) - E(S_i)E(S_j)$. The former term is the probability that both i and j get their own hats. There are 8! permutations in which i and j get their own hats, and 10! total permutations. Hence $E(S_iS_j) = \frac{8!}{10!}$. Thus,

$$\mathrm{Cov}(S_i, S_j) = E(S_iS_j) - E(S_i)E(S_j) = \frac{8!}{10!} - \frac{1}{100} = \frac{1}{90} - \frac{1}{100} = \frac{1}{900}.$$

(c)

$$E(S) = E(S_1+S_2+\ldots+S_{10}) = E(S_1)+E(S_2)+\ldots+E(S_{10}) = 10\times\frac{1}{10} = 1.$$

$$\mathrm{Var}(S) = \mathrm{Cov}(S, S) = \mathrm{Cov}(S_1 + S_2 + \ldots + S_{10}, S_1 + S_2 + \ldots + S_{10}).$$

By the bilinearity of covariance,

$$\mathrm{Cov}(S,S) = \sum_{i=1}^{10}\sum_{j=1}^{10} \mathrm{Cov}(S_i, S_j).$$

There are 100 terms in this sum. There are 10 terms for which $i = j$ and 90 terms for which $i \neq j$. The former terms are each equal to $\frac{9}{100}$ by (a) and the latter are equal to $\frac{1}{900}$ by (b). Hence, the total sum is $\frac{90}{100} + \frac{90}{900} = 1 = \mathrm{Var}(S)$.

3. (Ross 1.17) If $\mathrm{Cov}(X_i, X_j) = ij$, find

(a) $\mathrm{Cov}(X_1 + X_2, X_3 + X_4)$

(b) $\mathrm{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$.

ANSWER: use bilinearity of covariance to get
$\mathrm{Cov}(X_1 + X_2, X_3 + X_4) = \mathrm{Cov}(X_1, X_3 + X_4) + \mathrm{Cov}(X_2, X_3 + X_4) =$
$\mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_1, X_4) + \mathrm{Cov}(X_2, X_3) + \mathrm{Cov}(X_2, X_4) = 3+4+6+8 = 21$.
By similar arguments, part b is 54.

4. Four dice are rolled. Let X_i be the value of the ith roll. Let $Y = X_1X_2X_3X_4$ be the product of the rolls. Using the independence of the X_i to avoid long calculations, compute the mean and variance of Y.

ANSWER:

$$E(Y) = E(X_1X_2X_3X_4) = E(X_1)E(X_2X_3X_4) =$$

$$E(X_1)E(X_2)E(X_3X_4) = E(X_1)E(X_2)E(X_3)E(X_4) = (\frac{7}{2})^4.$$

Similarly,

$$E(Y^2) = E(X_1^2X_2^2X_3^2X_4^2) = \big(E(X_1^2)E(X_2^2)E(X_3^2)E(X_4^2)\big) =$$

$$\left(\frac{1+4+9+16+25+36}{6}\right)^4 = \left(\frac{91}{6}\right)^4.$$

Thus, $\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \left(\frac{91}{6}\right)^4 - (\frac{7}{2})^8$.

5. Three hundred people on an airplane are offered two dinner choices: grilled chicken and vegetarian lasagna. Suppose that everyone takes a dinner and each person independently chooses the chicken with probably .6 and the lasagna with probability .4. Let C be the number of people who choose chicken and L the number of people who choose lasagna. Calculate the following: $E(C)$, $E(L)$, $\mathrm{Var}(C)$, $\mathrm{Var}(L)$, $\mathrm{Cov}(C, L)$. Using the central limit theorem, estimate the amount of lasagna and chicken the airline should have on hand to ensure that there is at most a five percent chance that there will be an insufficient supply of either of the two items.

ANSWER: Let C_i be the function that is 1 if ith person gets chicken and zero otherwise. Let L_i be function that is 1 if the ith person gets lasagna and zero otherwise. So $L_i + C_i = 1$ for each $i \in \{1, 2, \ldots, 300\}$. Then for each i, we have $E(C_i) = .6$ and $E(L_i) = .4$ so $E(C) = 180$ and $E(L) = 120$. Also, for each i, we have $\mathrm{Var}(C_i) = E(C_i^2) - E(C_i)^2 = .6 - .26 = .24$ and $\mathrm{Var}(L_i) = E(L_i^2) - E(L_i)^2 = .4 - .16 = .4$. The variance of a sum of *independent* random variables is the sum of the variances. So $\mathrm{Var}(C) = 300\mathrm{Var}(C_i) = 72$ and $\mathrm{Var}(L) = 300\mathrm{Var}(L_i) = 72$. The standard deviation is $\sqrt{72} \sim 8.5$. With probability about .95, we will have C within two standard deviations of its mean, i.e., $180 - 17 \leq C \leq 180 + 17$. And since $L = 300 - C$, we have $120 - 17 \leq L \leq 120 + 17$. So if we have 197 chicken meals and 137 lasagna meals, there is an approximately ninety-five percent probability that there will be no shortage of *either one*. (If you momentarily forget about lasagna and only ask for the probability to be ninety-five percent that there is no *chicken shortage*, then you only need about 194 chicken meals. I also gave credit for this answer.)
Also, $\mathrm{Cov}(C, L) = \sum_{i=1}^{300} \sum_{j=1}^{300} \mathrm{Cov}(C_i, L_j)$. The terms in this sum are zero if $i \neq j$, by independence. But when $i = j$, we have $E[C_i, L_i] = 0$ and $E[C_i]E[L_i] = -.24$. Thus $\mathrm{Cov}(C_i, L_j) = -.24$. Summing up the 300 non-zero terms gives $-\mathrm{Cov}(C, L) = -72$. Now, the correlation coefficient is $-\frac{Cov(C,L)}{\sigma_C \sigma_L} = -72/(\sqrt{72}\sqrt{72}) = -1$. That is, C and L are maximally negatively correlated. (This makes sense since $C + L = 300$; so when C is high, L is low, and vice versa.)

6. (Ross 2.9) A model for the movement of a stock supposes that, if the present price of the stock is s, then—after one time period—it will be either us with probability p or ds with probability $1 - p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30 percent after the next 1000 time periods if $u = 1.012$, $d = .990$, and $p = .52$. (Hint: take logarithms and use

the central limit theorem.)

ANSWER: Let X_i be equal to 1.012 if the stock goes up at the ith step and .990 if the stock goes down at the ith step. We are interested in knowing the probability that $S(0)X_1X_2X_3 \ldots X_{1000}$ is at least equal to $1.30S(0)$. This is the same as the probability that the product $X_1X_2X_3 \ldots X_{1000}$ is at least 1.30, which is in turn the same as the probability that $\log X_1 + X_2 + \ldots + X_{1000} \leq \log 1.3$. For each $1 \leq i \leq 1000$, write $Y_i = \log X_i$. Then each Y_i is a random variable. We have $E[Y_i] = .52\log(1.012) + .48(\log .990) = .0013787$. Also $E[Y_i^2] = .52(\log 1.012)^2 + .48(\log .990)^2 = .000122476$. Thus $\text{Var}(Y_i) = E(Y_i^2) - E(Y_i)^2 = .000120575$.
Write $Y = \log X = \sum_{i=1}^{1000} Y_i$. Then $E(Y) = 1000E(Y_i) = 1.3787$, and $\text{Var}(Y) = 1000\text{Var}(Y_i) = .120575$. Thus, the standard deviation of Y is $\sigma_Y = .347239$. Now, $\log 1.3 = .262363$. And $\frac{E(Y)-\log 1.3}{\sigma_Y} = 3.21488$. In other words, $\log 1.3$ is 3.21488 standard deviations of Y below the expected value of Y. The central limit theorem says that the probability that that Y is less than $\log 1.3$ is approximately the same as the probability that a standard normal variable is less than -3.21488. Consulting a normal distribution table, we see that this probability is about .00065. The answer to the original question is thus about $1 - .00065$.

7. (Ross 1.13) Let $X_1, \ldots X_n$ be independent random variables, all having the same distribution with expected value μ and variance σ^2. The random variable $\overline{X}$, defined as the arithmetic average of these variables, is called the *sample mean*. That is, the sample mean is given by

$$\frac{\sum_{i=1}^n X_i}{n}.$$

(a) Show that $E[\overline{X}] = \mu$.

(b) Show that $\text{Var}[\overline{X}] = \sigma^2/n$.

The random variable S^2, defined by

$$S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1},$$

is the *sample variance*. (Denominator is $n-1$, not n, due to (d).)

(c) Show that $\sum_{i=1}^n (X_i - \overline{X})^2 = \sum_{i=1}^n X_i^2 - n\overline{X}^2$.

(d) Show that $E[S^2] = \sigma^2$.

ANSWER:

(a)

$$E[\overline{X}] = E[\sum_{i=1}^{n} \frac{X_i}{n}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}(nu) = u$$

(b)

$$\begin{aligned}
\mathrm{Var}[\overline{X}] &= E[\overline{X}^2] - E[\overline{X}]^2 \\
&= E[\sum_{i=1}^{n} X_i^2 - E[\sum X = \overline{X}_i]^2 \\
&= \frac{1}{n^2}\mathrm{Var}[\sum_{i=1}^{n} X_i] \\
&= \frac{1}{n^2}(n\sigma^2) \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

(c)

$$\begin{aligned}
\sum_{i=1}^{n}(X_i - \overline{X})^2 &= \sum_{i=1}^{n}[X_i^2 - 2X_i\overline{X} + \overline{X}^2] \\
&= \sum_{i=1}^{n} X_i^2 - 2\sum_{i=1}^{n} X_i\overline{X} + \sum_{i=1}^{n}\overline{X}^2 \\
&= \sum_{i=1}^{n} X_i^2 - (2\overline{X})\overline{X} + \overline{X}^2 \\
&= \sum_{i=1}^{n} X_i^2 - 2\overline{X}^2 + \overline{X}^2 \\
&= \sum_{i=1}^{n} X_i^2 - \overline{X}^2
\end{aligned}$$

(d) Using part (c), we have

$$
\begin{aligned}
E[S^2] &= E\left[\frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1}\right] = \\
&= E\left[\frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1}\right]
\end{aligned}
$$

Now, using previous parts, we see that for each i, $\sigma^2 = \mathrm{Var}(X_i^2) = E[X_i^2] - E[X_i]^2$. Thus, $E[X_i^2] = \sigma^2 + \mu^2$. Also, $\frac{\sigma^2}{n} = \mathrm{Var}[\overline{X}] = E[\overline{X}^2] - E[\overline{X}]^2 = E[\overline{X}^2] - \mu^2$. Thus, $E[\overline{X}^2] = \frac{\sigma^2}{n} + \mu^2$. Plugging these values for $E[\overline{X}^2]$ and $E[X^2]$ into the expression for $E[S^2]$, we compute

$$
\begin{aligned}
E[S^2] &= E\left[\frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1}\right] = \\
&= E\left[\frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1}\right] \\
&= \frac{\sum_{i=1}^n (\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2)}{n-1} \\
&= \frac{n\sigma^2 + n\mu^2 - \sigma^2 + n\mu^2}{n-1} \\
&= \sigma^2
\end{aligned}
$$