Mathematics of Finance
Problem Set 1
Due September 21

1. (Like Ross, 1.7) Two cards are randomly selected from a deck of 52 playing cards. What is the probability that they are both aces? What is the conditional probability that they are both aces, given that they are of different suits?

2. Suppose 10 people toss their hats into a box and then take turns drawing hats at random, so each person ends up with one hat. How many possible permutations of the hats are there? Assume that each such permuation is equally likely. Let S_i be one if the ith person gets her own hat, zero otherwise.

(a) Calculate mean and variance of S_i.

(b) Calculate the covariance of S_i and S_j when $i \neq j$.

(c) Let S be the number of people who get their own hat. Calculate the mean and variance of S.

3. (Ross 1.17) If $\text{Cov}(X_i, X_j) = ij$, find

(a) $\text{Cov}(X_1 + X_2, X_3 + X_4)$

(b) $\text{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$.

4. Four dice are rolled. Let X_i be the value of the ith roll. Let $Y = X_1X_2X_3X_4$ be the product of the rolls. Using the independence of the X_i to avoid long calculations, compute the mean and variance of Y.

5. Three hundred people on an airplane are offered two dinner choices: grilled chicken and vegetarian lasagna. Suppose that everyone takes a dinner and each person independently chooses the chicken with probably .6 and the lasagna with probability .4. Let C be the number of people who choose chicken and L the number of people who choose lasagna. Calculate the following: $E(C)$, $E(L)$, $\text{Var}(C)$, $\text{Var}(L)$, $\text{Cov}(C, L)$. Using the central limit theorem, estimate the amount of lasagna and chicken the airline

should have on hand to ensure that there is at most a five percent chance that there will be an insufficient supply of either of the two items.

6. (Ross 2.9) A model for the movement of a stock supposes that, if the present price of the stock is s, then—after one time period—it will be either us with probability p or ds with probability $1-p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30 percent after the next 1000 time periods if $u = 1.012$, $d = .990$, and $p = .52$. (Hint: take logarithms and use the central limit theorem.)

7. (Ross 1.13) Let $X_1, \ldots X_n$ be independent random variables, all having the same distribution with expected value μ and variance σ^2. The random variable $\overline{X}$, defined as the arithmetic average of these variables, is called the *sample mean*. That is, the sample mean is given by

$$\frac{\sum_{i=1}^{n} X_i}{n}.$$

(a) Show that $E[\overline{X}] = \mu$.

(b) Show that $\mathrm{Var}[\overline{X}] = \sigma^2/n$.

The random variable S^2, defined by

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1},$$

is the *sample variance*. (Denominator is $n-1$, not n, due to (d).)

(c) Show that $\sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n} X_i^2 - n\overline{X}^2$.

(d) Show that $E[S^2] = \sigma^2$.