Limit Theorems
Final Exam
Due 10:30 a.m., December 16

1. Prove that an infinitely divisible probability distribution μ on $\mathbb{R}$ is supported on the rational numbers $\mathbb{Q} \subset \mathbb{R}$ (i.e., $\mu(\mathbb{Q}) = 1$) if and only if some translation of μ by a rational number has the form $e_\lambda(M)$ for some $\lambda > 0$ and some probability measure M which is supported on $\mathbb{Q}$.

2. Let $\{X_i\}$ be a sequence of i.i.d. bounded random variables taking values in the integer grid $\mathbb{Z}^2$. Let $S_n = \sum_{i=1}^n X_i$. Prove that the sequence S_n is a recurrent Markov chain if and only if $\mathbb{E}[X_1] = 0$. Is this still true if we allow the X_i to be unbounded?

3. Can you give an example of a sequence of probability measures μ_n on $\mathbb{R}$ whose characteristic functions ϕ_n converge point-wise (as n tends to infinity) to the function $1_\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers? What if we replace $1_\mathbb{Z}$ with 1_A, where A is the set of integers whose absolute values are perfect squares?

4. Let μ be a probability measure on a measure space $(\Omega, \mathcal{F})$. Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are σ subalgebras of $\mathcal{F}$ and that Q_a is a family of measures on $(\Omega, \mathcal{F})$ (indexed by $a \in \Omega$) that gives a regular conditional probability for μ given $\mathcal{A}$. That is, for all $S \in \mathcal{F}$, the maps $a \to Q_a(S)$ are $\mathcal{A}$ measurable; $Q_x(A) = 1_A$ for all $A \in \mathcal{A}$; and for all $\mathcal{F}$-measurable functions f, it is μ-almost everywhere the case that $\mathbb{E}^\mu[f|\mathcal{A}] = \int f dQ_a$. Suppose further that the following hold:

(a) $\mathcal{A}$ is μ-independent of $\mathcal{B}$ (i.e., for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\mu(A \cap B) = \mu(A)\mu(B)$)

(b) $\mathcal{A}$ is μ-independent of $\mathcal{C}$

(c) For μ-almost all a, the algebras $\mathcal{B}$ and $\mathcal{C}$ are Q_a-independent of each other.

Prove that $\mathcal{A}$ is μ-independent of the σ algebra generated by $\mathcal{B} \cup \mathcal{C}$. Is this still true without requirement (c)?

5. Let Ω be a (infinite dimensional with a countable basis) separable Hilbert space and let $\mathcal{B}$ be the smallest field (not necessarily a σ-field) containing all subsets of Ω of the form $\{f : (f, g) \in A\}$ where $g \in \Omega$ and A is a Borel subset of $\mathbb{R}$. Let $\mathcal{F}$ be the σ-field generated by $\mathcal{B}$.

(a) Is it possible to construct a finitely additive (though not necessarily σ-additive) probability measure on $(\Omega, \mathcal{B})$ that is invariant under all Hilbert space automorphisms of Ω (besides the trivial measure which assigns measure 1 to the origin)? [Hint: try a Gaussian measure. Recall: a Hilbert space automorphism T is a linear bijection from the Hilbert space to itself that preserves the inner product, i.e., $(f, g) = (Tf, Tg)$).]

(b) Is it possible to construct a σ-additive probability measure on $(\Omega, \mathcal{F})$ that is invariant under all Hilbert space automorphisms of Ω (besides the trivial measure which assigns measure 1 to the origin)?