Mathematics of Finance
Final Preparation
December 19

To be thoroughly prepared for the final exam, you should

1. know how to do the homework problems.
2. be able to provide (correct and complete!) definitions for the following words (listed in no particular order) and understand what they mean: variance, expectation, covariance, correlation coefficient, empirical variance (sample variance), random variable, American and European put and call options, stock, bond, annuity, perpetuity, martingale, risk-neutral probability, normal random variable, independent, conditional probability, arbitrage, replicating portfolio, present value, binomial tree model, marking to market, short selling, solvency assumption, divisibility assumption, no-arbitrage principle, central limit theorem, optional stopping time theorem, fundamental theorem of asset pricing, CAPM, efficient frontier, market portfolio, portfolio, unit vector, minimum variance portfolio, beta, Black-Scholes formula for European calls and puts, Greek parameters (vega, delta, gamma, theta, rho), lognormal distribution, forward, future, dividend, market maker, bid, ask, bid-ask spread, in the money, out of the money, at the money, Brownian motion, marking to market, geometric Brownian motion, diversifiable risk, idiosyncratic risk, volatility, implied volatility, historic volatility, regression line, risk premium.

Below are some examples of problems that could be on the final:

1. If elected, Candidate A would implement a budget with 10 million dollars in taxes and 9 million in government spending. Candidate B would implement a budget with 10 million dollars in taxes and 11 million dollars in government spending. Candidate C would implement a budget with 8 million dollars in taxes and 12 million dollars in government spending. Suppose that all three candidates have a one in three chance of winning the election. Let T be the amount of taxes in the budget and S the amount of spending.

(a) Compute the expectation of T.

(b) Compute the covariance of S and T.

(c) Compute the variance of S

(d) Compute the variance of T

(e) Compute the correlation coefficient of S and T.

(f) Show how to compute the variance of the deficit $S - T$ from the variance of S, the variance of T, and the covariance of S and T.

2. Consider the binomial tree model with $S(0) = 1$, $u = .01 + .0005$, and $d = -.01 + .0005$, $r = .0002$, and $p = 1/2$. Compute the following:

1. The expected value of $S(252)$.
2. The median value of $S(252)$.
3. The variance of $S(252)$.
4. The annual volatility σ.
5. The risk neutral probability p_* that the stock goes up at a given step.
6. What fraction of wealth should an investor put in the risky asset in order to maximize the expected log return during one time step?

Use the central limit theorem to approximate:

1. The probability that $S(252) > 1$.
2. The risk neutral probability that $S(252) > 1$.
3. The market price of a contract that pays 1 at time 252 if $S(252) > 1$ and zero otherwise.

You may write your answers in terms of the function
$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$

3. Compute the present value of an annuity that pays 10 every year for five years, starting exactly one year from now. Assume that the interest rate is .05.

4. Define these words, in about two to four sentences each, giving relevant formulas when you can.

1. implied volatility
2. correlation coefficient
3. efficient frontier
4. risk neutral probability
5. martingale
6. lognormal distribution
7. future
8. forward

5. Describe the primary assumptions and conclusions of mean-variance portfolio optimization theory and the Capital Asset Pricing Model.

6. Suppose the last eight closing prices of a stock have been 100, 101, 102, 103, 101, 100, 102, 101. Compute the seven-day historical daily volatility. Use this to find the seven-day historical annual volatility σ.

7. Suppose there are three stocks whose random returns over some fixed time period (perhaps a decade) have covariance matrix

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and expected return vector

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

(a) Compute the minimum variance portfolio.

(b) Compute the market portfolio.

(c) Sketch the efficient frontier.

(d) Suppose that in addition there is a risk-free asset with return .5. Sketch the efficient frontier in this case.

You may use the fact that

$$C^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & .5 \end{pmatrix}$$

8. Let $f(x) = \begin{cases} x & x \leq 10 \\ 2x - 10Q + rf & 10 < x < 20 \\ 4x - 50 & x \geq 20 \end{cases}$. Sketch a graph of $f(x)$. Let $S(T)$ denote the value of a stock at time T. Describe a portfolio made up positive quantities of European calls and puts (with exercise time T and strike prices that you specify) such that the value of the portfolio at time T is $f(S(T))$.

9. Suppose that X is a normal random variable with expectation 5 and standard deviation 4. Compute the expectation of e^X.

10. Give Black Scholes formula for European call options with strike price X, interest rate r, current stock price $S(0)$, expiry date T, and volatility σ.

(a) What is the price (in terms of the other parameters) of a European put option when $X = 0$. How about an American put option with strike price $X = 0$? Give a simple explanation for your answers.

(b) According to the Black-Scholes formula, what is the price (in terms of the other parameters) of a European put option when $\sigma = 0$?

(c) Using the answer to (b), what is the formula for the vega of this option at time zero in terms of the parameters given above? (You may leave this as a derivative; i.e., you don't have to explicitly compute the derivative or simplify your answer.)

(d) In general, will the price of a call option be higher or lower or the same if the stock is expected to pay dividends before time T? Explain your answer.

11. Let $g(X) = C_X^E(0)$ be the price of a European call option on a stock with fixed expiration time $T = 1$ and strike price X, where

$$\begin{cases} g(X) = .095(X-5)^2 & 0 \leq X \leq 5 \\ 0 & X \geq 5 \end{cases}$$

Assume that an option can be bought or sold at this price for any value of X. Suppose that r is the interest rate on a riskless asset and $e^{-rT} = .95$.

(a) Compute the risk neutral probability density function for the price $S(T)$ of the stock at time T.

(b) Assuming no arbitrage is possible, compute the price of a contract that pays 1 if $1 \le X \le 2$ and 0 otherwise.

12. Given a sequence of independent fair coin tosses, write $X_i = 1$ if the ith toss comes up heads and -1 if it comes up tails. Which of the following are martingales? (Justify your answers.)

(a) $S(0) = 0$ and $S(n) = \sum_{i=1}^{n} X_i$ for $n \ge 1$.

(b) $S(0) = 0$ and $S(n) = \prod_{i=1}^{n} X_i$ for $n \ge 1$.

(c) $S(0) = 0$ and $S(n)$ is defined recursively by $S(n) = S(n-1) + X_n(S(n-1) + 1)$.

13. In the Black-Scholes framework, if $\sigma = .2$, $r = .05$, and $S(0) = 1$, then the risk neutral probability distribution of $\log S(5)$ is a Gaussian.

(a) Compute its mean and variance.

(b) Using the answer to (a), give a formula for the price of a derivative that pays $D(5) = f(S(5))$ at time 5, where $f(x) = \cos(x)$. (You don't have to actually evaluate the integral.)

14. You look up the prices of European calls on a particular stock and discover the following prices for December, 2006 calls:

STRIKE PRICE	BID	ASK
195	47.50	47.80
200	45.60	46
205	43.80	44.20
210	42.30	42.70

A friend has confidence that this stock will do well in the future. This friend offers to bet any amount of money you like at even odds that the stock will be worth at least 200 on the expiry date in December, 2006 (which we suppose is exactly one year from now). Can you combine a bet with this friend with buying and selling call options and/or riskless assets (at interest rate .05 per year) in order to make a risk-free profit with zero initial investment?

15. If the β for Cisco stock is 3 and the market portfolio has an expected annual return of .11 while the riskless asset has an expected annual return of .05, what does the CAPM model predict as the expected annual return of Cisco stock? What is the "risk premium" in this setting?

16. When the strike price and expiry date are the same, which should be priced higher, an American put or a European put? Ignoring the possibility of dividends, which should be priced higher, a portfolio with one American call and minus one American put or a portfolio with one European call and minus one European put. Explain your answers.