**Mathematics of Finance Final**
**December 19**
**100 points, 120 minutes**

1. (25 points) Define these terms in about three or four sentences each. Give complete and correct definitions.

(a) efficient frontier (in mean-variance portfolio optimization theory)

(b) systemic risk of a portfolio

(c) risk neutral probability that an event $A$ occurs at time $T$

(d) market portfolio (as in CAPM theory)

(e) replicating portfolio

2. (15 points) In the Black-Scholes framework, if $\sigma = .4$, $r = .04$, and $S(0) = 5$, then the risk neutral probability distribution of $\log S(10)$ is a Gaussian.

(a) Compute its mean and variance.

(b) Suppose that $\mu$ is the mean computed in (a). In terms of $\mu$, give the price at time 0 of a derivative that pays 1 at time 10 if $S(10) > e^{\mu}$ and 0 otherwise.

3. (5 points) Suppose the last four closing prices of a stock have been 10, 11, 12.10, 13.31. Compute the three-day historical daily volatility.

4. (15 points) Consider four stocks, and let $S_i(n)$ denote the value of the $i$th stock at time $n$. Suppose that $S_1(0) = S_2(0) = S_3(0) = S_4(0) = 10$, but that $S_1(1)$, $S_2(1)$, $S_3(1)$, and $S_4(1)$ are independent random variables and

$S_1(1)$ is equal to 16 with probability .5 and 6 with with probability .5

$S_2(1)$ is equal to 17 with probability .5 and 7 with probability .5

$S_3(1)$ equal to 18 with probability .5 and 8 with probability .5.

$S_4(1)$ equal to 19 with probability .5 and 9 with probability .5.

Consider the time period from zero to one and compute the following:

(a) The covariance matrix $C$ and expectation vector $m$ for the returns of these four assets during the time interval from 0 to 1.

(b) The minimum variance portfolio.

(c) The market portfolio, assuming that the risk free rate of return is zero.

5. (20 points) Suppose that for all $X \geq 0$ the price of a European call option on a stock with fixed expiration time $T = 4$ and strike price $X$ is given by $g(X) = e^{-X-.2}$. (Assume that options can be both bought and sold at these prices for any $X$.) Suppose that the risk free interest rate is $r = .05$.

(a) Compute the risk neutral probability density function $p$ for the price $S(4)$ of the stock at time 4.

(b) Assuming no arbitrage is possible and that there are no dividends, compute $S(0)$. (If you did not solve (a), I will give full credit for a correct formula involving the density function $p$.)

(c) If the stock *does* pay dividends (but the call prices are still as given above), will the value of $S(0)$ be higher or lower than the answer given in (b)? Explain.

(d) Assuming no arbitrage is possible, compute the price of a contract that pays 1 at time $T = 4$ if $0 \leq S(4) \leq 5$ and zero otherwise. (If you did not solve (a), I will give full credit for a correct formula involving the density function $p$.)

6. (5 points) If the historical $\beta$ of a stock is 1.5 and the market portfolio has an expected annual return of .12 while the riskless asset has an expected annual return of .04, what does the CAPM model predict as the expected annual return of the stock?

7. (5 points) CAPM predicts a straight-line relationship between which two of the following four aspects of a portfolio? Diversifiable risk, non-diversifiable risk, volatility, expected return.

8. (10 points) An individual with wealth of fifty thousand dollars and a utility function $U(x) = \log(x)$ is given the opportunity to bet an amount $X$ of her choosing on a fair coin toss. The bet is in her favor in that if the coin comes up heads, she wins $2X$ and if it comes up tails she only loses $X$. How much should she bet in order to maximize her expected utility?