18.440 PROBLEM SET FOUR, DUE OCTOBER 7

A. FROM TEXTBOOK CHAPTER FOUR:

1. Problems: 23, 30, 50, 57.
2. Theoretical Exercises: 13, 19, 23, 28.

B. Define the covariance $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$.

1. Check that $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$, that $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$, and that $\mathrm{Cov}(\cdot, \cdot)$ is a bilinear function of its arguments. That is, if one fixes one argument then it is a linear function of the other. For example, if we fix the second argument then for real constants a and b we have $\mathrm{Cov}(aX + bY, Z) = a\mathrm{Cov}(X, Z) + b\mathrm{Cov}(Y, Z)$.
2. If $\mathrm{Cov}(X_i, X_j) = ij$, find $\mathrm{Cov}(X_1 - X_2, X_3 - 2X_4)$.
3. If $\mathrm{Cov}(X_i, X_j) = ij$, find $\mathrm{Var}(X_1 + 2X_2 + 3X_3)$.

C. Instead of maximizing her expected wealth $E[W]$, Jill maximizes $E[U(W)]$ where $U(x) = -(x - x_0)^2$ and x_0 is a large positive number. That is, Jill has a *quadratic utility function*. (It may seem odd that Jill's utility declines with wealth once wealth exceeds x_0. Let us assume x_0 is large enough so that this is unlikely.) Jill currently has W_0 dollars. You propose to sample a random variable X (with mean μ and variance σ^2) and to give her X dollars (she will lose money if X is negative) so that her new wealth becomes $W = W_0 + X$.

1. Show that $E[U(W)]$ depends on μ and σ^2 (but not on any other information about the probability distribution of X) and compute $E[U(W)]$ as a function of x_0, W_0, μ, σ^2.
2. Show that given μ, Jill would prefer for σ^2 to be as small as possible. (One sometimes refers to σ as *risk* and says that Jill is *risk averse*.)
3. Suppose that $X = \sum_{i=1}^{n} a_i X_i$ where a_i are fixed constants and the X_i are random variables with $E[X_i] = \mu_i$ and $\mathrm{Cov}[X_i, X_j] = \sigma_{ij}$. Show that in this case $E[U(W)]$ depends on the μ_i and the σ_{ij} (but not on any other information about the joint probability distributions of the X_i) and compute $E[U(W)]$. Hint: first compute the mean and variance of X.
4. Read the Wikipedia article on "Modern Portfolio Theory". Summarize what you learned in two or three sentences.