What is a random surface?

Scott Sheffield

Massachusetts Institute of Technology

July 3, 2022

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The presentation is designed for an online audience.

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You won't believe what happened next!

Frenkel: When people say “I hate math” what you’re really saying is, “I hate the way mathematics was taught to me.” Imagine an art class in which they only teach you how to paint a fence or wall but never show you the paintings of the great masters. Then of course years later you’re going to say, “I hate art....”

Colbert: But in math don't I have to know a fair amount of high end math to appreciate the work of the masters? It's almost as if you could show me a painting by a master but I don't have eyeballs yet. Don't you need to grow the math eyeballs to see the equations as beautiful?

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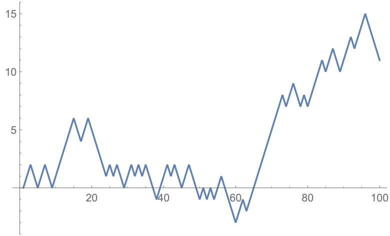
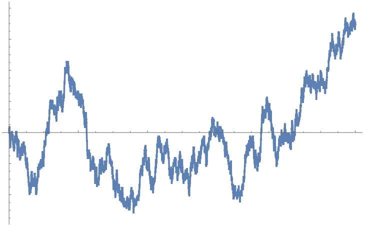
Google *Scott Sheffield* for homepage with these slides plus code for figures.

When student asks: what's the "canonical" random path?

INSTRUCTOR: Consider the simple random walk on $\mathbb{Z}$. At each time step a coin toss decides whether position goes up or down. If you shrink the graph horizontally by a factor of C and vertically by a factor of $\sqrt{C}$, then the $C \to \infty$ limit is a random path called *Brownian motion* (a random function from $\mathbb{R}_+$ to $\mathbb{R}$).





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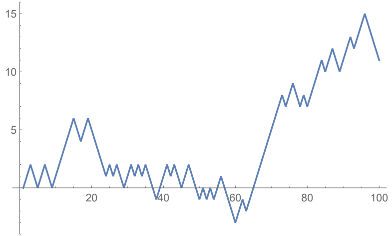

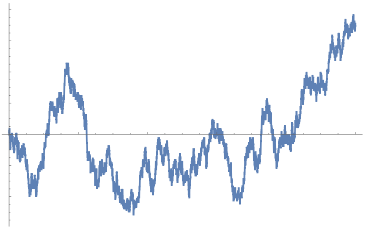

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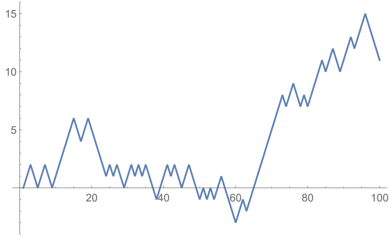

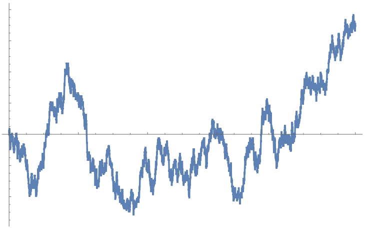

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INSTRUCTOR: Sure! Fix $0 = t_0 < t_1 < \ldots < t_n$. Specify the joint law of $B(t_1), \ldots, B(t_n)$ by making increments $B(t_k) - B(t_{k-1})$ independent normal random variables with mean 0, variance $t_k - t_{k-1}$. Extend to countable dense set (Kolmogorov extension), then all t (Kolomogorov-Čentsov).

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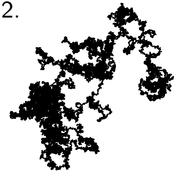
INSTRUCTOR: Use a vector $\Big(B_1(t), B_2(t), \ldots, B_d(t)\Big)$ of independent Brownian motions. For example, here's a Brownian loop (Brownian motion conditioned to return to origin) in the case $d = 2$.



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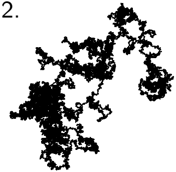

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The student is happy. Now imagine a similar dialog for random surfaces.

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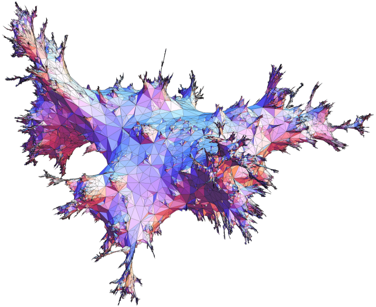
INSTRUCTOR: Take a uniformly random triangulation of sphere with n triangles: i.e., among *all* ways to glue n triangles along boundaries to make a topological sphere, choose *one* at random. Here's a 30,000-triangle example by Budzinski given a 3D "spring embedding." The $n \to \infty$ limit is a random fractal surface called the *Brownian sphere*. Also a *peanosphere*, a *pure Liouville quantum gravity sphere* and a *conformal field theory*.



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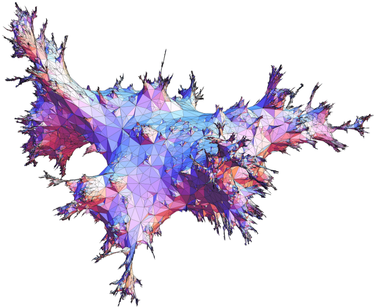

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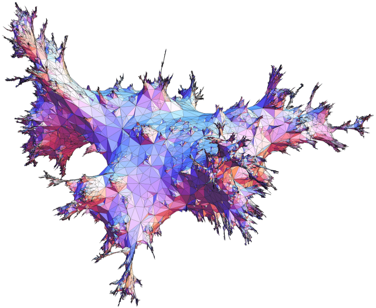

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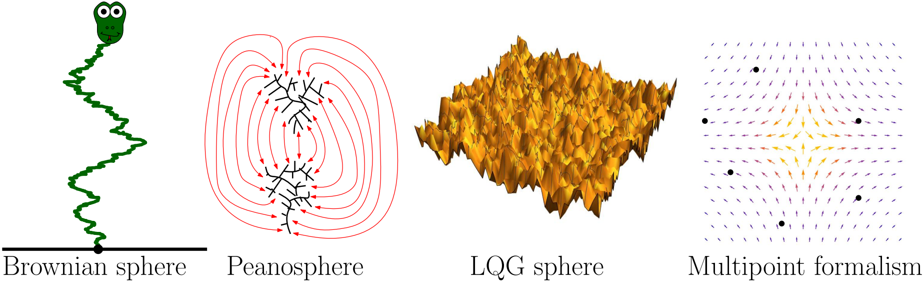
INSTRUCTOR: They all are! The difference comes down to the features of the limit we keep track of. View them as different aspects of the same universal object. Four blind mathematicians feel the surface of an elephant and describe four things:



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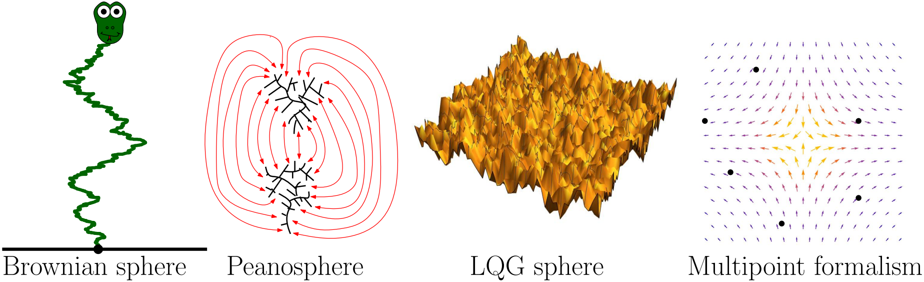

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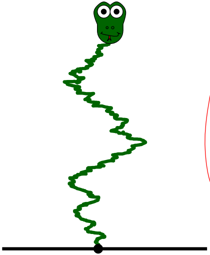

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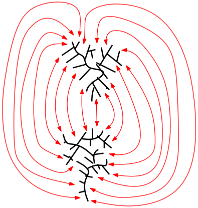

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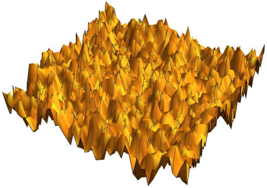
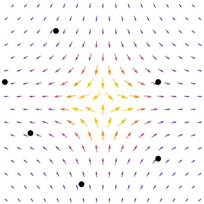


LQG sphere



Multipoint formalism

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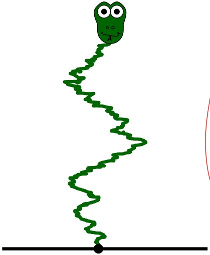

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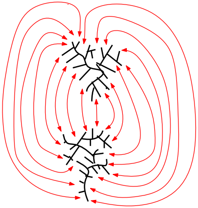

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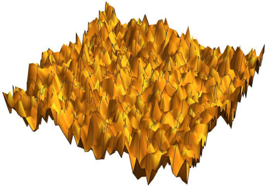

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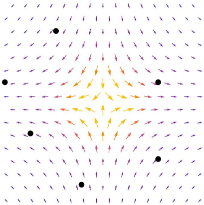

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4. **Conformal field theory:** a *collection of multipoint functions* representing (regularized) integrals of products of the form $\prod e^{\alpha_i \phi(x_i)}$ w.r.t. a certain infinite measure. The infinite measure is the *Polyakov measure* which is the product of an unrestricted-area measure on LQG spheres (with defining field ϕ) and Haar measure on the Möbius group $\mathrm{PSL}(2,\mathbb{C})$ (to select an embedding in $\mathbb{C}$).

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Exponent motivated by discrete models: number of triangulations with n faces and k marked points scales like $C\beta^n n^{-7/2+k}$ for model-dependent constants C and β. Natural to weight the counting measure by β^{-n} so we are left with power-law decay. Unrestricted-area discrete measure (appropriately rescaled) converges to the measure above as area-per-triangle ϵ goes to zero.

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Off-critical case: If we replace β by the "off-critical" $\beta(1+\epsilon\mu)$ then the limit is $A^{-7/2+k}e^{-\mu A}dA$, which is finite if $\mu > 0$ and $k \geq 3$. The $e^{-\mu A}$ factor is common in physics formulations, e.g. Polyakov's early work where μ is called the *cosmological constant* and motivated by *Liouville's equation*.

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The surfaces are "rougher" for large d, "smoother" for small d, converging to the Euclidean sphere as $d \to -\infty$. Defined as random metric spaces for any $d \leq 25$, but only finite-diameter/finite-volume if $d \leq 1$.

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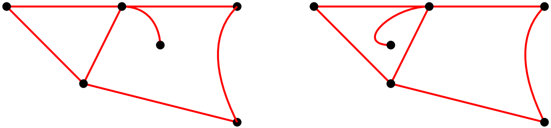
STUDENT: I'm getting lost. Can you give the four definitions you promised?

1. BROWNIAN SPHERE: A RANDOM-METRIC-SPACE LIMIT OF RANDOM PLANAR MAPS

Planar maps and Tutte's enumeration

Planar map: finite graph embedded in plane, where two embeddings are equivalent if an orientation-preserving homeomorphism of $\mathbb{C} \cup \{\infty\}$ takes one to other. Figures below isomorphic as graphs but represent different planar maps.



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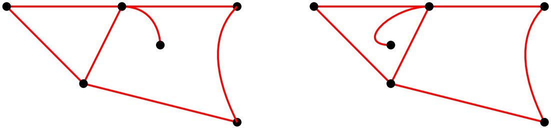

Combinatorics: planar map determined by graph plus "clockwise cyclic ordering" of the edges surrounding each vertex. Number of planar maps with n edges is finite.

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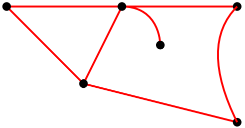

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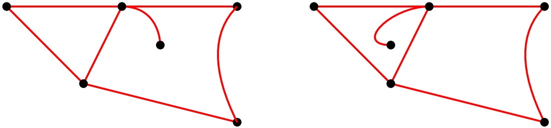

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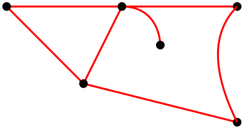

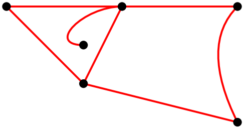

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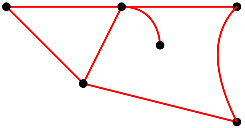

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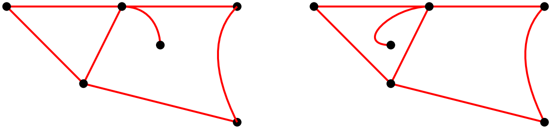

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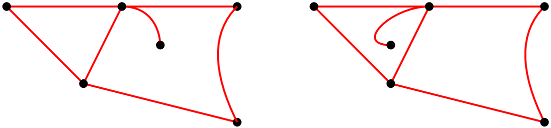

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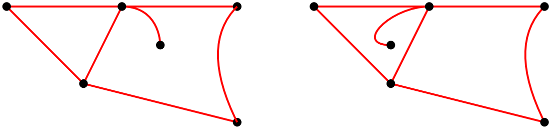

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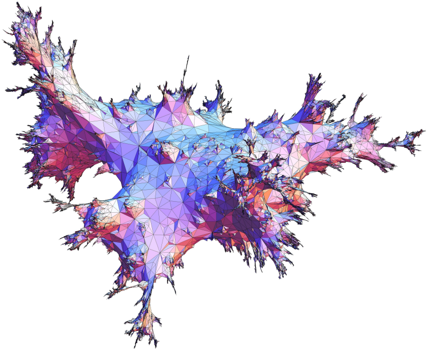
Brownian sphere: metric space limit of random planar map



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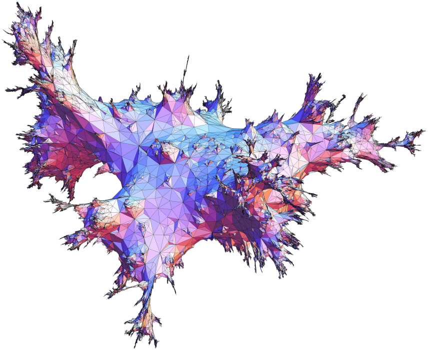

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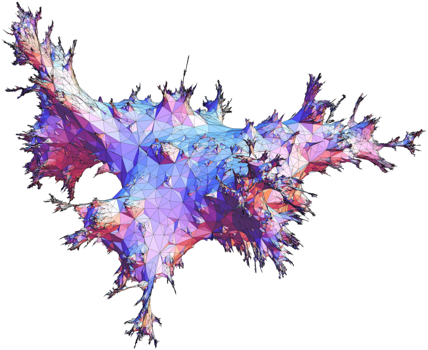

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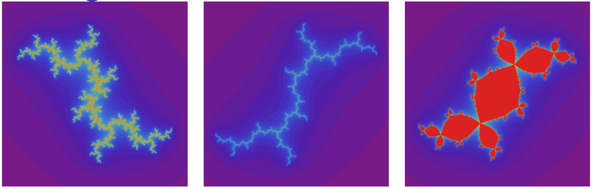
Benedikt Stufler's simulations: color vertices by their mean distance to others
`https://www.dmg.tuwien.ac.at/stufler/gabmanim.html`

2. PEANOSPHERE: A MATING OF RANDOM TREES

Warmup: matings of deterministic fractal trees



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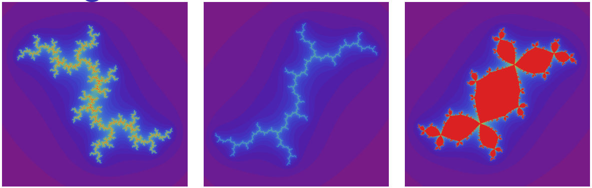

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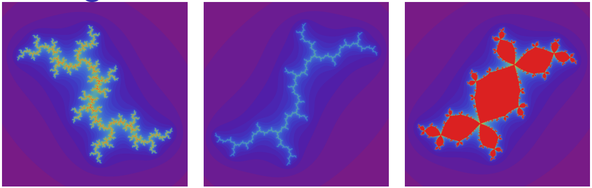

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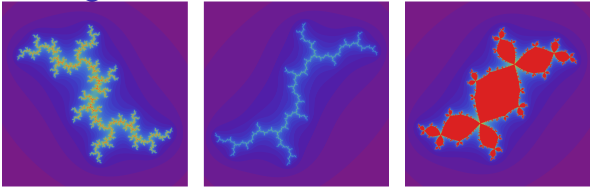

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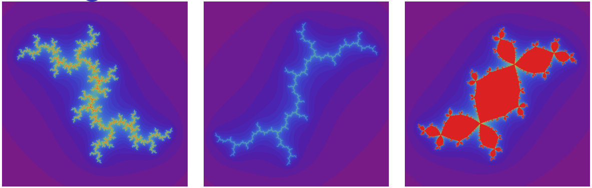

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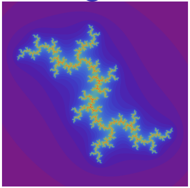
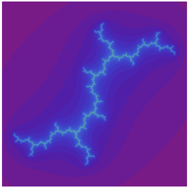
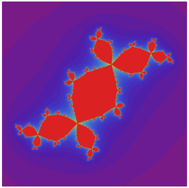

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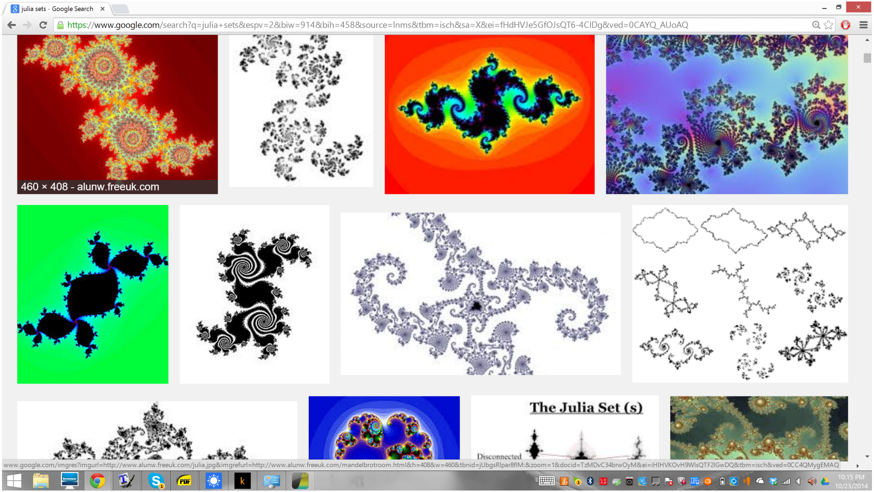
Arnaud Chéritat's simulations:

`https://www.math.univ-toulouse.fr/~cheritat/MatMovies/`

Google search for Julia sets



```
c = I; S = 2000; A = Table[0, {j, 1, S}, {k, 1, S}]; For[i = 0,  i < S, i++;
 For[j = 0, j < S, j++; count = 0; x = 3 (i I + j)/S - 1.5 - 1.5 I;
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```
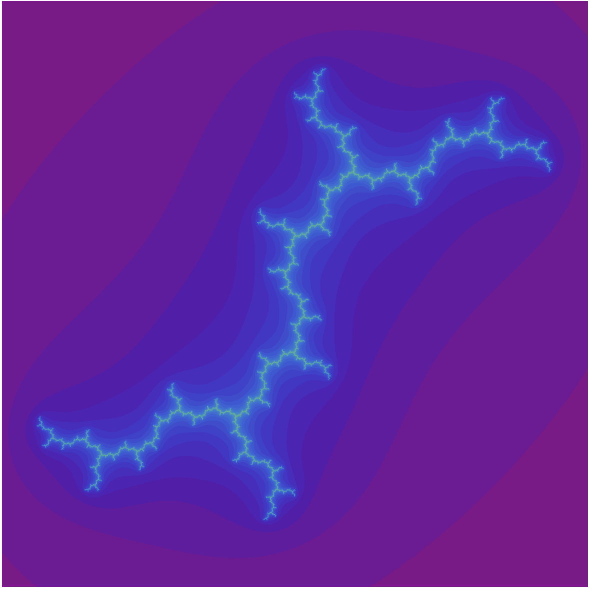
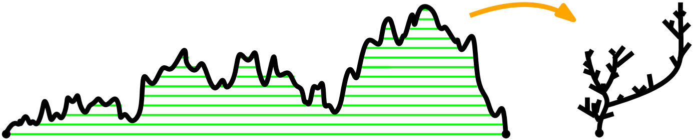



Student asks: what's the "canonical random tree"?



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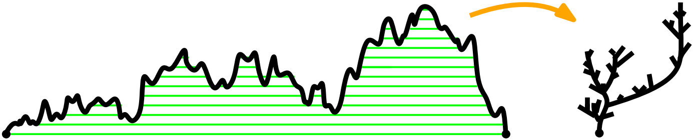

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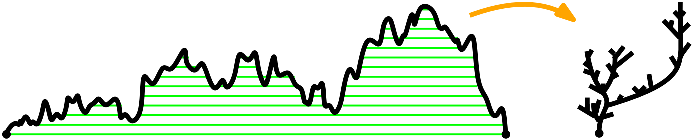

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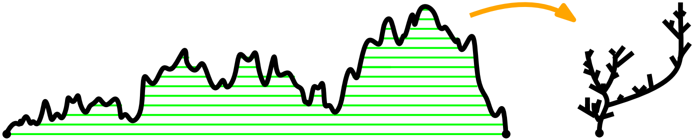

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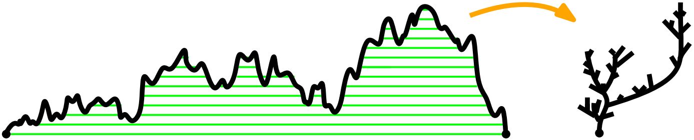

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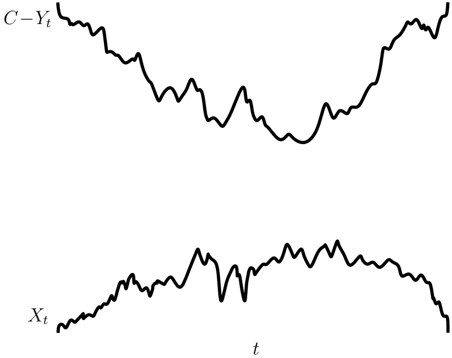
Brownian tree is the (limiting) "uniformly random planar tree" of a given size.

MATING RANDOM TREES

X, Y independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of X and $C - Y$ are disjoint.



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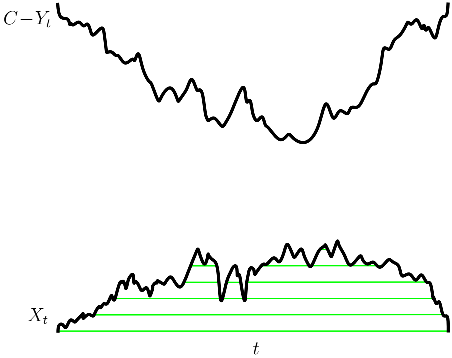

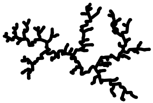

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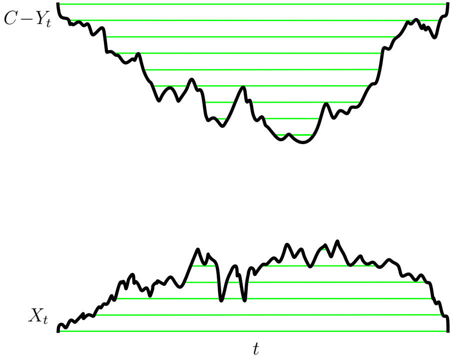

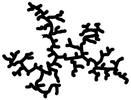

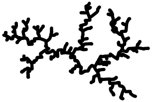

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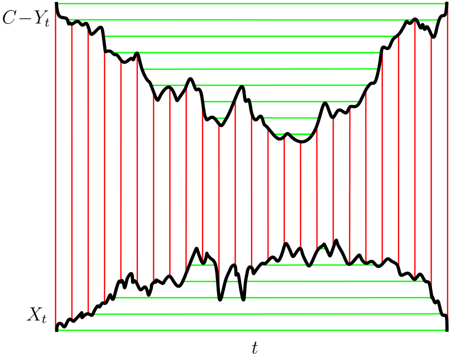

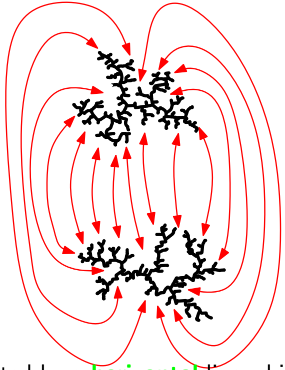

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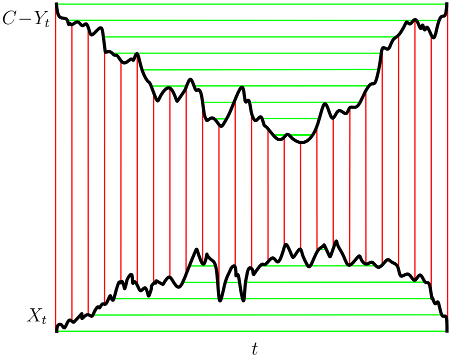

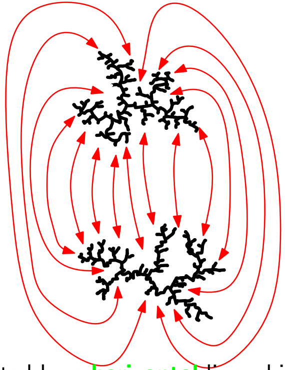

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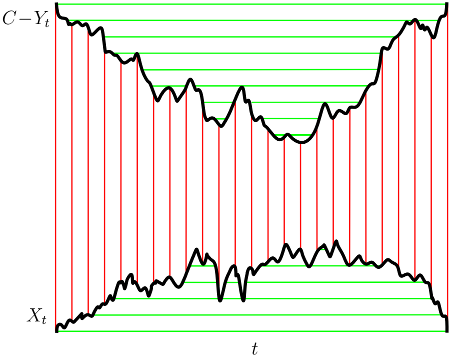
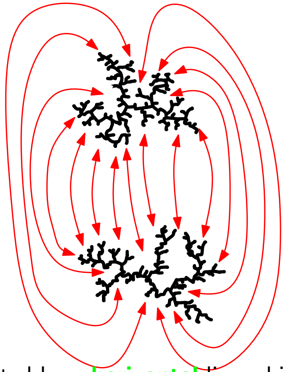
Q: What is the resulting structure?

MATING RANDOM TREES

X, Y independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of X and $C - Y$ are disjoint.





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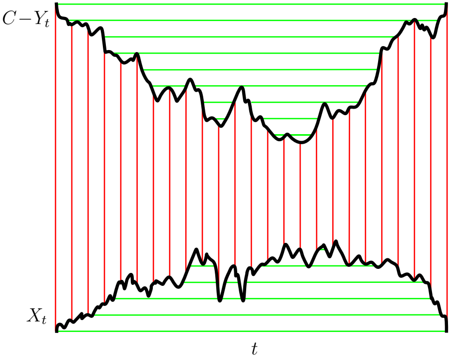

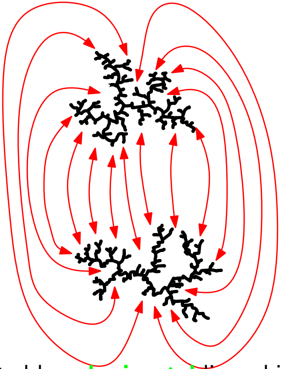

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Surface is topologically a sphere by Moore's theorem

Theorem (Moore 1925)

Let $\cong$ be any topologically closed equivalence relation on the sphere S^2. Assume that each equivalence class is connected and not equal to all of S^2. Then the quotient space $\mathrm{S}^2/\cong$ is homeomorphic to S^2 if and only if no equivalence class separates the sphere into two or more connected components.

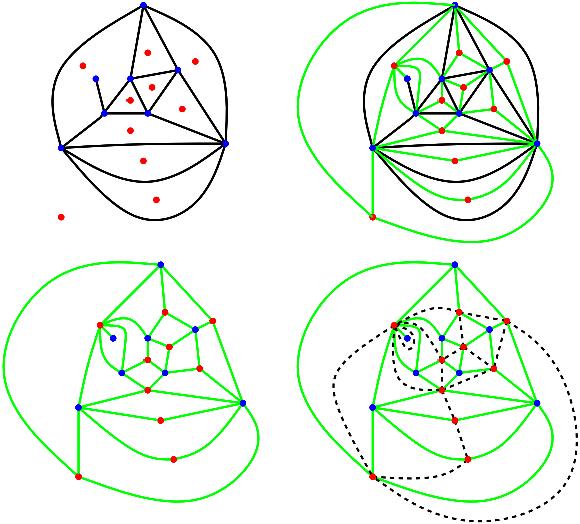
An equivalence relation is topologically closed iff for any two sequences (x_n) and (y_n) with

$x_n \cong y_n$ for all n

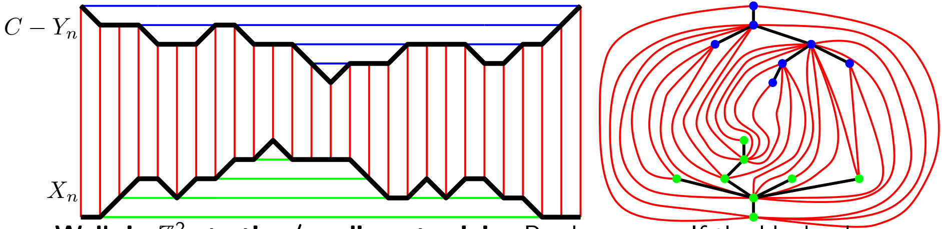
$x_n \to x$ and $y_n \to y$

we have that $x \cong y$.

Correspondence: quadrangulations and planar maps



Discrete version of mating of trees: Mullin's bijection



Walk in $\mathbb{Z}^2$ starting/ ending at origin: Produce map. If the black edges are erased, one is left with a red *quadrangulation* $\mathcal{Q}$.

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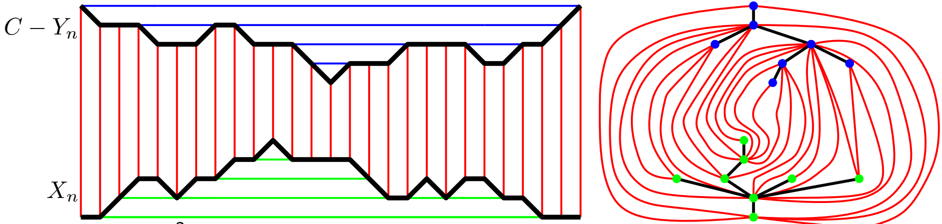

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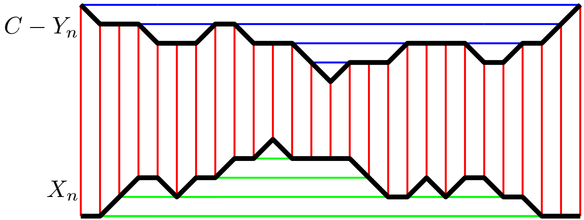

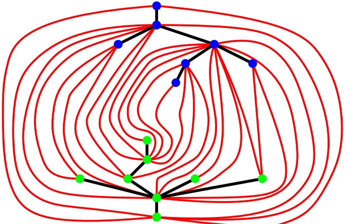

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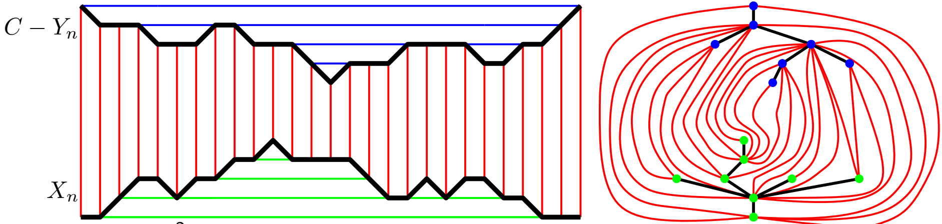

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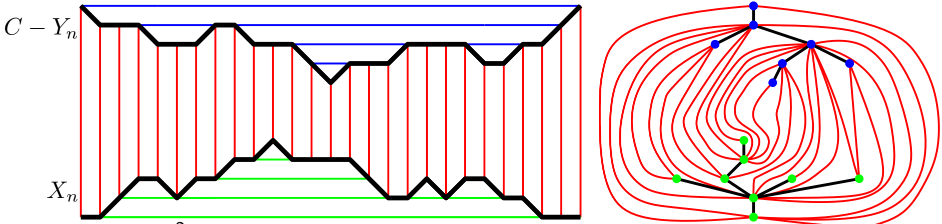

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Random (X_n, Y_n) yields *random* (M, T). $P(M) \sim \#$ spanning trees of M.

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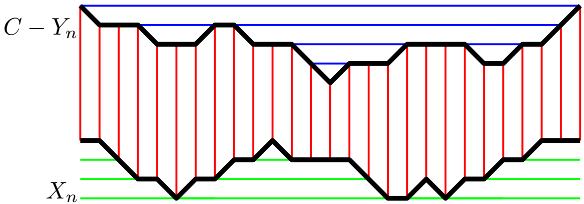
Weirdly... if (M, T) is tree decorated random surface, the law of M is kind of a like "law of surface embedded in $\mathbb{R}^d$ with $d = -2$."

Unconstrained variant

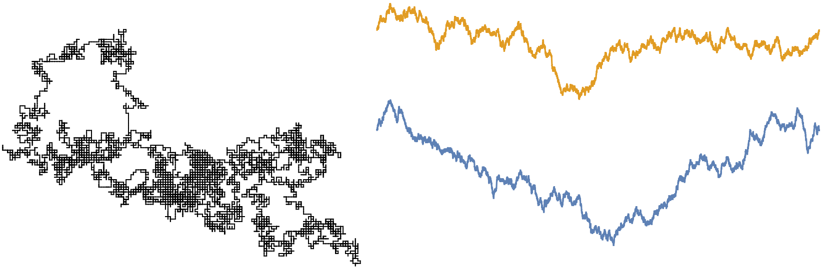
We remark that there is a variant of the Mullin bijection in which we relax the restriction that X_n is non-negative, see below.



Here we can imagine that the left and right sides of the above rectangle are glued to one another (so that both X_n and Y_n then become indexed by a circle).

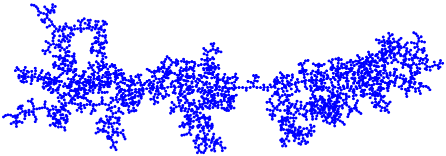
2D walk (X_t, Y_t), coordinates $X(t)$, $C - Y(t)$.

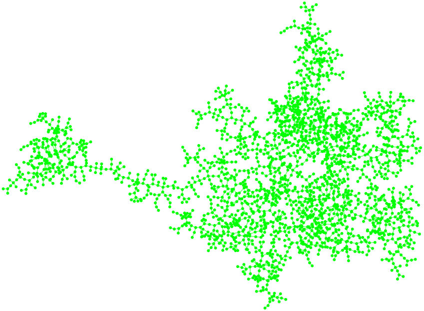
```
n=2000;Z=Table[0,{j,1,n+1}];A=Z;B=Z;For[j=1,j<n,++j,A[[j+1]]=A[[j]]+If[2 RandomReal[]>1+A[[j]]/(n-j),1,-1];B[[j+1]]=B[[j]]
+If[2 RandomReal[]>1+B[[j]]/(n-j),1,-1]]; X=n/2+(A+B)/2;Y=n/2+(A-B)/2;{ListPlot[{X,n+Sqrt[n]-Y},PlotJoined->True,Axes->False],
Graphics[Table[Line[{{X[[j]],Y[[j]]},{X[[j+1]],Y[[j+1]]}}],{j,1,n}]]}
```



The corresponding pair of trees

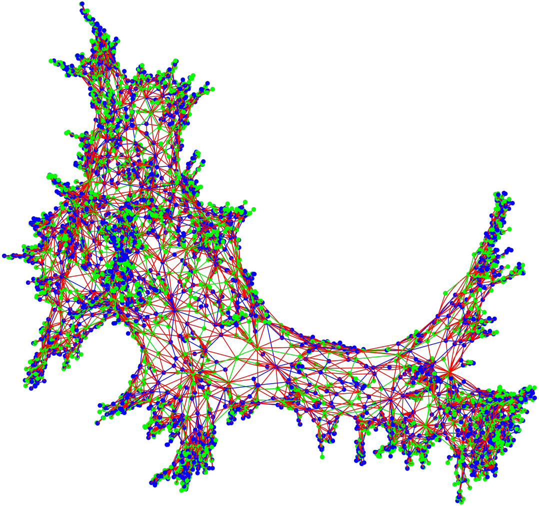
```
vertnumX=Z+1;vertnumY=Z+1;last=Z+1;minxloc=1;minyloc=1; For[j=1,j<n+1,++j, If[X[[j]]<X[[minxloc]],minxloc = j];
 If [Y[[j]] < Y[[minyloc]], minyloc = j]]; count = 1;
For[ j = minxloc, j < n + 1, ++j,If[X[[j+1]]>X[[j]],vertnumX[[j+1]]=++count;last[[X[[j+1]]]] = count, vertnumX[[j+1]]=
 last[[X[[j+1]]]]]]; vertnumX[[1]] = vertnumX[[n + 1]];
For[j = 1,j<minxloc-1,++j,If[X[[j+1]]>X[[j]],vertnumX[[j+1]] =++count;last[[X[[j + 1]]]] = count,  vertnumX[[j + 1]] =
 last[[X[[j + 1]]]]]]; vertnumY[[minyloc]] = ++count; last = Z + count; For[j = minyloc, j < n + 1, ++j,
 If[Y[[j + 1]] > Y[[j]], vertnumY[[j + 1]] = ++count; last[[Y[[j + 1]]]] = count,  vertnumY[[j + 1]] = last[[Y[[j + 1]]]]]];
 vertnumY[[1]] = vertnumY[[n + 1]]; For[j = 1, j < minyloc - 1, ++j,  If[Y[[j + 1]] > Y[[j]], vertnumY[[j + 1]] = ++count;
  last[[Y[[j + 1]]]] = count, vertnumY[[j + 1]] = last[[Y[[j + 1]]]]]];
{GraphPlot[SimpleGraph[Table[Style[vertnumX[[j]] <-> vertnumX[[j + 1]], Blue], {j,1,n}]], VertexStyle -> Blue, GraphLayout ->
 {"SpringEmbedding"}], GraphPlot[SimpleGraph[Table[Style[vertnumY[[j]] <-> vertnumY[[j + 1]], Green], {j,1,n}]],
 VertexStyle -> Green, GraphLayout -> {"SpringEmbedding"}]}
```





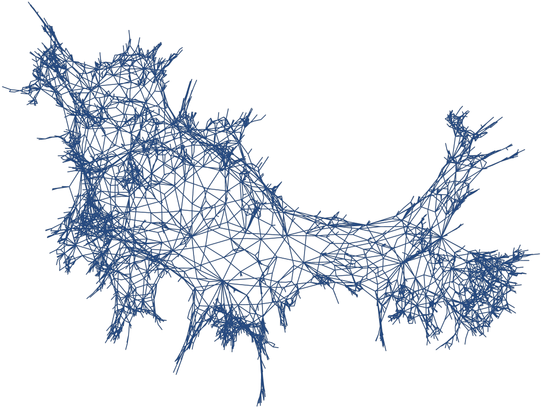
Add red edges connecting two trees to make planar map

```
g=Table[0->0,{j,1,3n/2}];count=1;For[j=1,j<=n,++j,g[[count++]]=Style[vertnumX[[j]]<->vertnumY[[j]],Red]];For[j=1,j<=n,++j,
If[X[[j]]<X[[j+1]],g[[count++]]= Style[vertnumX[[j]] <-> vertnumX[[j + 1]], Blue]]];For[j=1, j <= n,++j, If[Y[[j]]<Y[[j + 1]],
g[[count++]]=Style[vertnumY[[j]]<->vertnumY[[j+1]],Green]]];
GraphPlot3D[g, VertexStyle->Table[i ->If[i<vertnumY[[minyloc]],Blue,Green],{i,1,n/2+2}],GraphLayout ->{"SpringElectricalEmbedding"}]
```



Remove trees, show just quadrangulation (red edges)

```
M=SparseArray[Table[0,{j,1,n/2+2},{k,1,n/2+2}]];For[j=1,j<n+1,j++,M[[vertnumX[[j]],vertnumY[[j]]]]+= 1];
x=GraphPlot3D[M,GraphLayout -> {"SpringElectricalEmbedding"},VertexShapeFunction->None]
```



Typing code below after making 3D figure makes animated spinning version.

```
ResourceFunction["ExportRotatingGIF"]["C:\\filename.gif", %, ImageSize -> 1200]
```

Changing the correlation parameter

Correlation: Instead of taking X_t and Y_t to be independent Brownian motions, we can make them *correlated*. Varying the correlation coefficient ρ (between -1 and 1) gives a 1 parameter family of random surfaces.

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`https://www.normalesup.org/~bettinel/simul_FK.html`

1̃. BROWNIAN SPHERE: A MATING OF A DIFFERENT PAIR OF RANDOM TREES RELATED TO BROWNIAN SNAKE

Cori-Vauquelin-Schaeffer bijection helps us enumerate rooted maps M (or rooted quadrangulations $\mathcal{Q}$) instead of (M, T) pairs. Similar to the Mullin bijection but with a few key differences.

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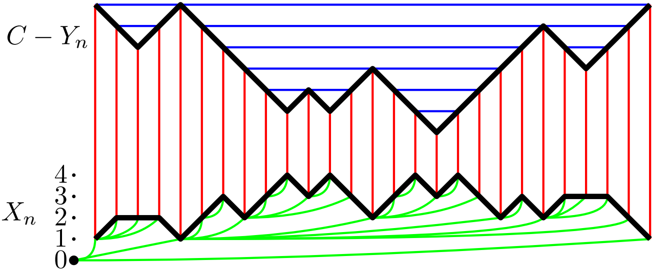
1. Instead of requiring (X_n, Y_n) to traverse lattice edges we at each step allow Y_n to change by ± 1 and X_n by either 0 or ± 1.
2. Instead of perfectly horizontal green chords, we draw chords that are one unit higher on the right than on the left. We draw one such chord leftward starting at each vertex on the graph of X_n, which means that we have to add an extra vertex of minimal height as shown.



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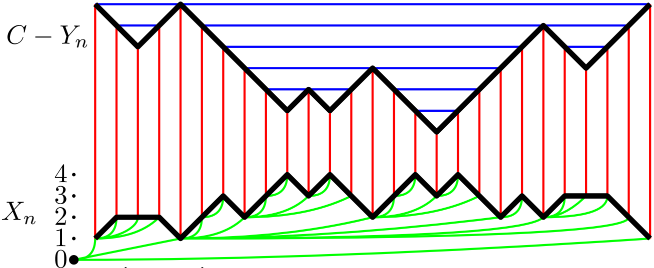
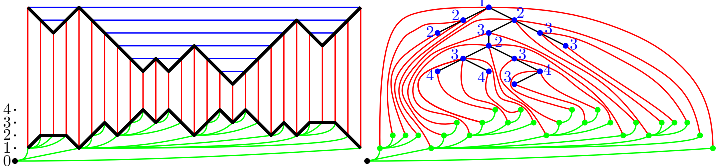
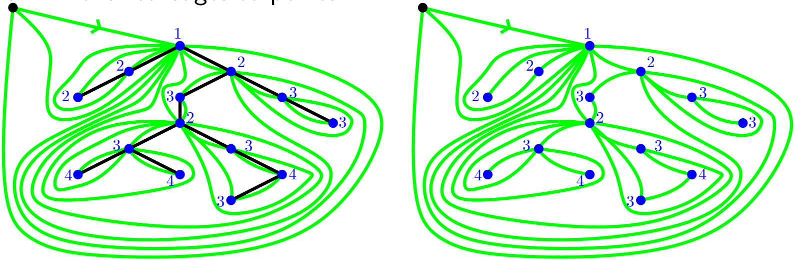


3. We consider only (X_n, Y_n) pairs for which the above picture has a special property: namely, whenever two red vertical lines are incident to the same blue chord, their lower endpoints have the *same height*.

Collapse blue to make tree. Condition 3 says two red edges starting at same blue vertex have same green vertex height—label each red vertex by that height.

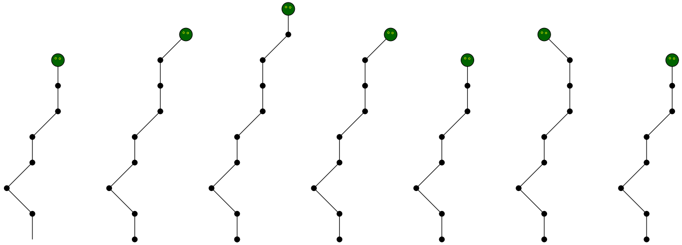


Then shrink the red edges to points.

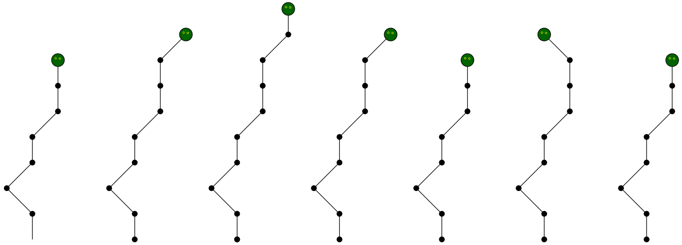


The construction above yields a bijection between

1. **Well-labeled rooted planar trees** (T, ℓ). Here ℓ maps vertices of T to positive integers; root has label 1, adjacent vertices differ by 0 or ± 1.
2. **Rooted quadrangulations** Q.

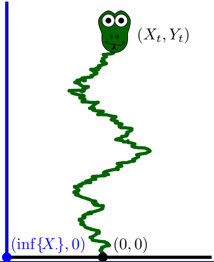


Markovian discrete snake: Condition on head height Y_k staying non-negative with $Y_0 = Y_{2n} = 0$. Let X_n be horizontal coordinate.



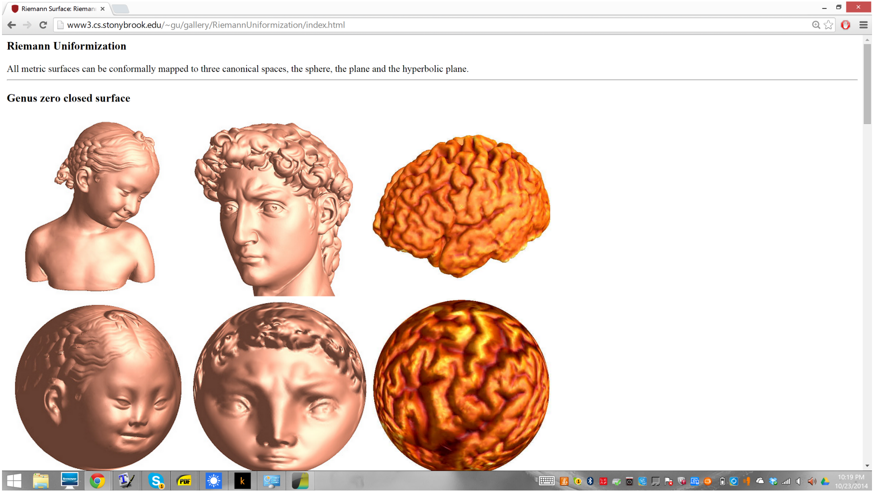
Markovian discrete snake: Condition on head height Y_k staying non-negative with $Y_0 = Y_{2n} = 0$. Let X_n be horizontal coordinate.

Brownian snake: Rescaling gives continuum Brownian snake (process by Le Gall in 1990's, term coined by Dynkin and Kuznetsov).



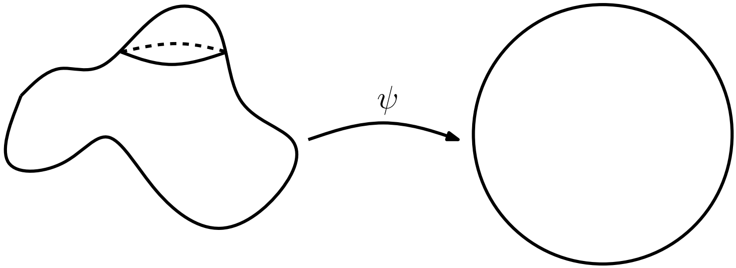
3. DEFINING THE LQG SPHERE USING THE GAUSSIAN FREE FIELD

Conformal maps (from David Gu's web gallery)



Picking a surface at random in the continuum

Uniformization theorem: every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere S^2 in R^3



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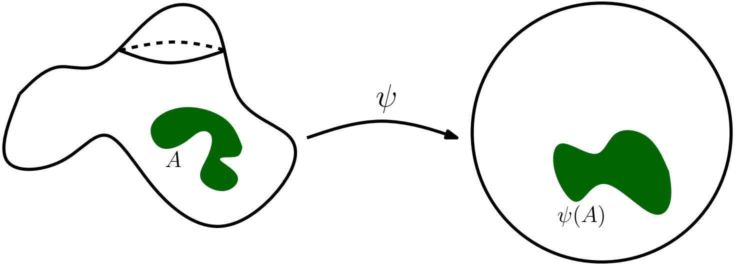

Isothermal coordinates: Metric for the surface takes the form $e^{\rho(z)}dz$ for some smooth function ρ where dz is the Euclidean metric.

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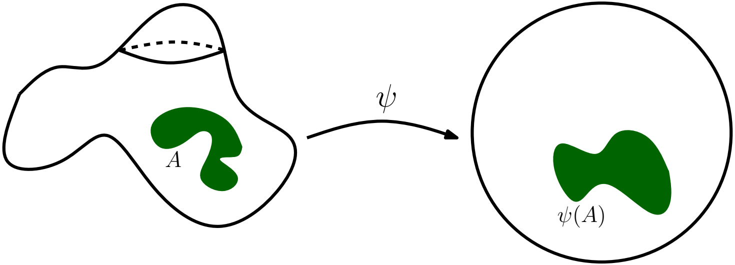

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$\Rightarrow$ Can parameterize the space of surfaces with smooth functions.

If $\rho = 0$, get the same surface

If $\Delta\rho = 0$, i.e. if ρ is harmonic, the surface described is flat

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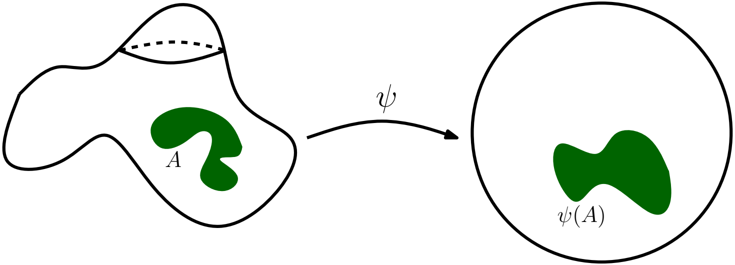

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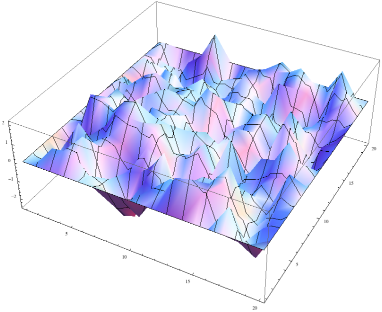
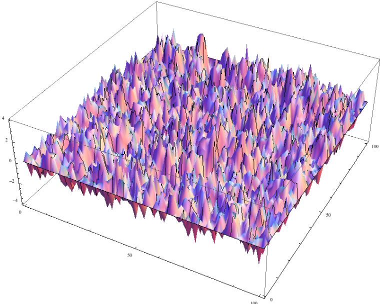
Question: Which measure on ρ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.





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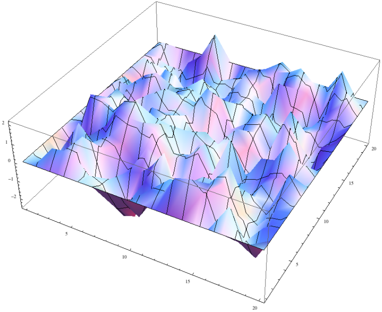
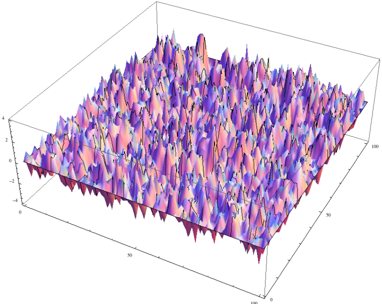
Measure on functions $\phi\colon D \to \mathrm{R}$ for $D \subseteq \mathrm{Z}^2$ and $\phi|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathrm{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (\phi(x) - \phi(y))^2 \right)$$





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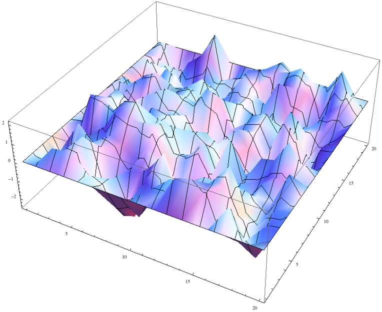
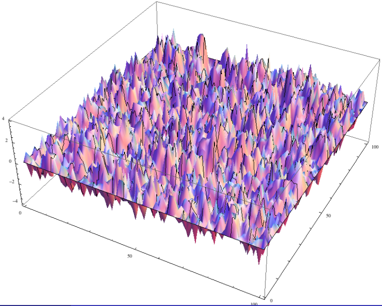
Natural perturbation of a harmonic function





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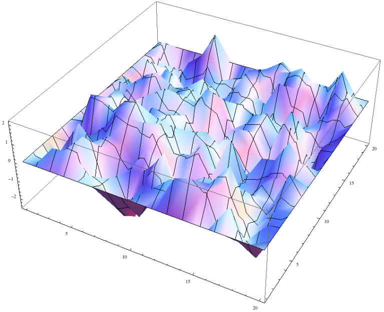
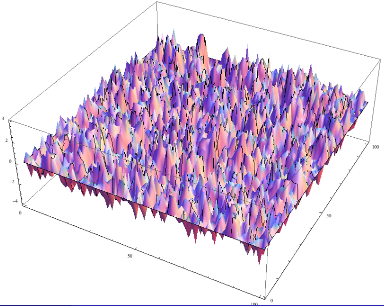
Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the **Dirichlet inner product**

$$(f,g)_\nabla = \frac{1}{2\pi}\int \nabla f(x)\cdot\nabla g(x)dx.$$





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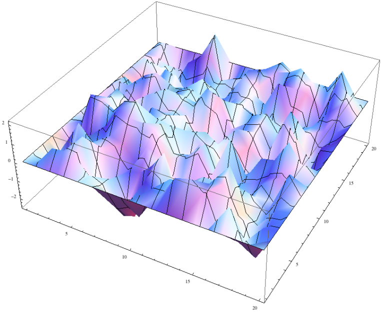

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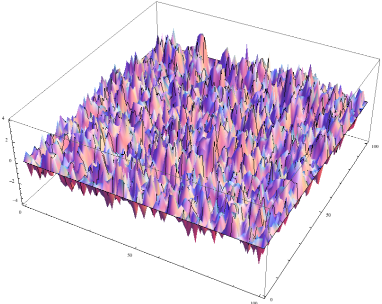
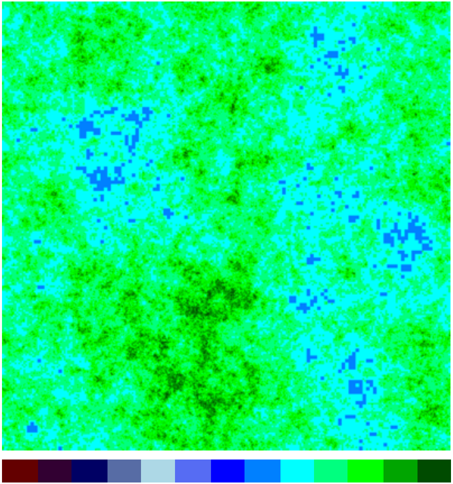


Continuum GFF not a function — only a generalized function

Liouville quantum gravity: $e^{\gamma\phi(z)}dz$
where ϕ is a kind of GFF and $\gamma \in [0, 2)$

$\gamma = 0.5$



(Number of subdivisions)

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Random surface model: Polyakov, 1980. Motivated by string theory.

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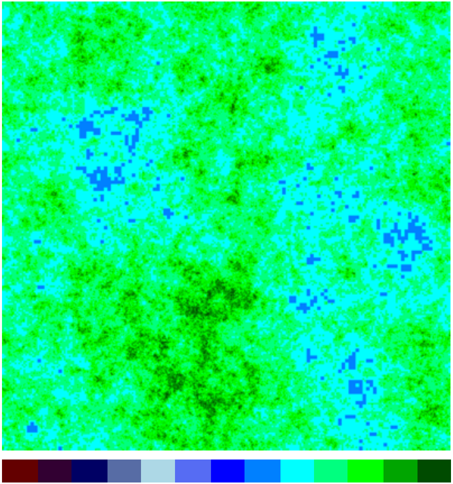

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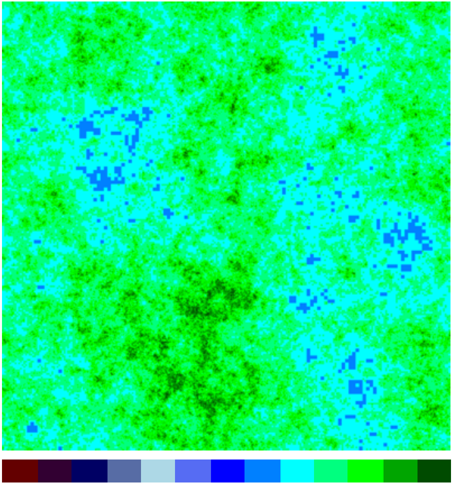

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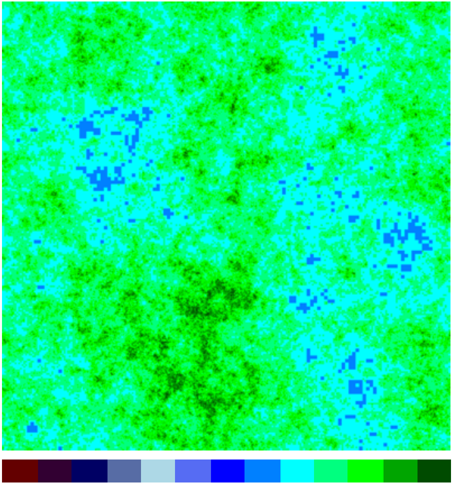

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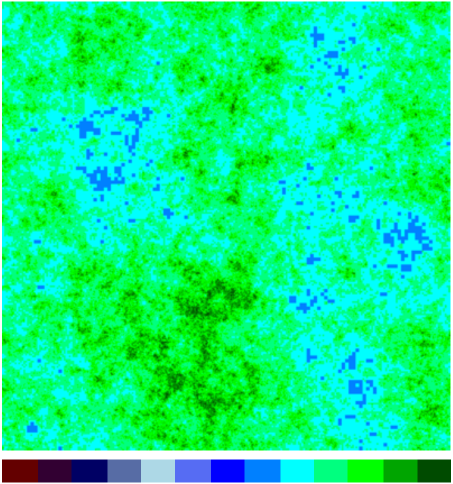
Figure: draw square blocks that are "about same size" w.r.t. this measure.

$\gamma = 1.0$



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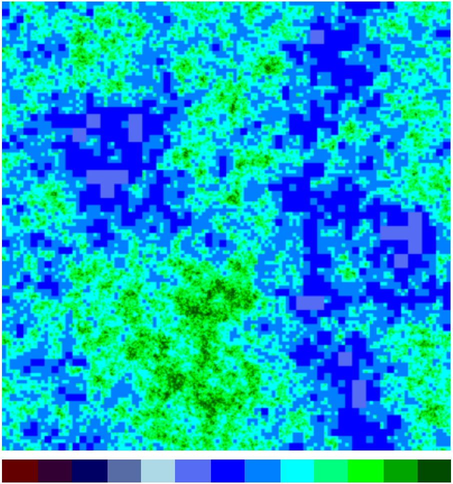

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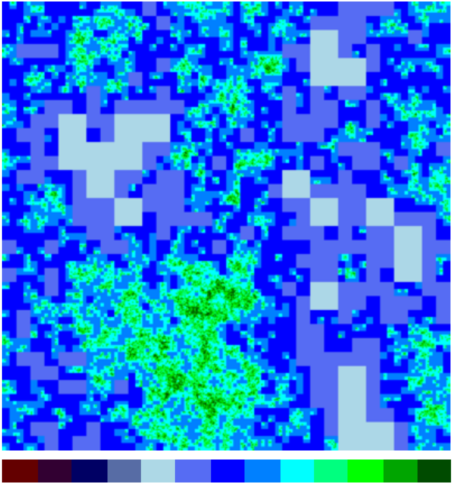
Figure: draw square blocks that are "about same size" w.r.t. this measure.

$\gamma = 2.0$



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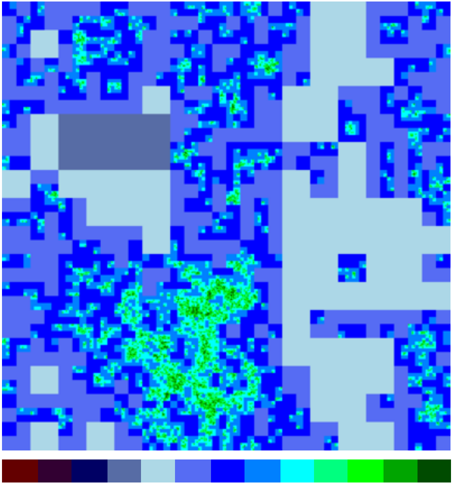
Does not make literal sense as ϕ takes values in the space of distributions.



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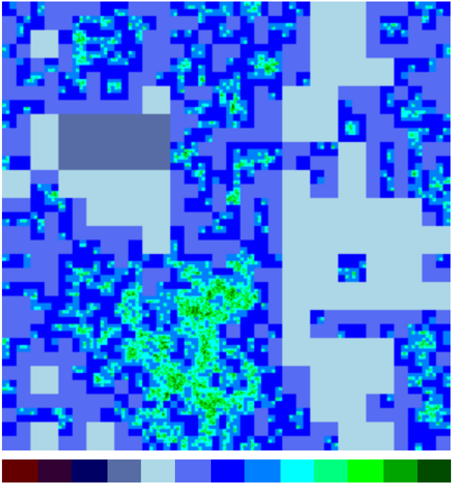
Can make sense of random area measure using a regularization procedure.



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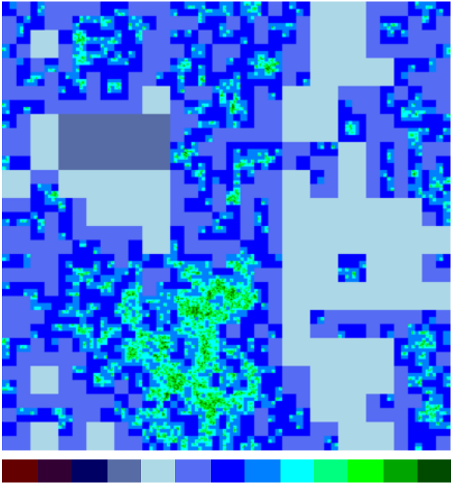

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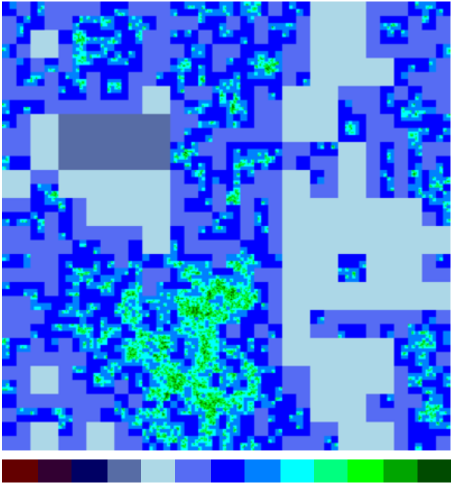
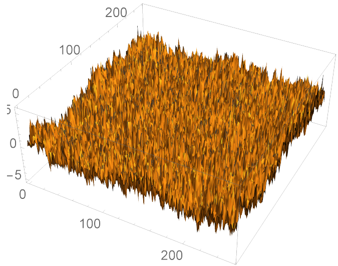
Areas of regions and lengths of curves are well defined.

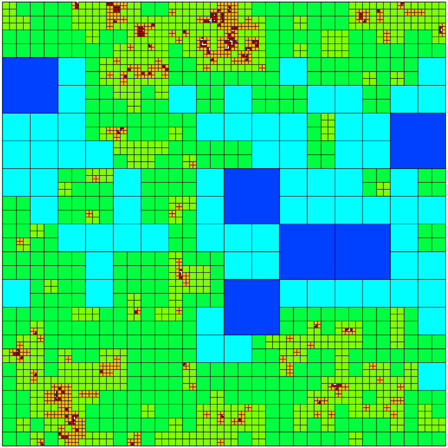


(Number of subdivisions)

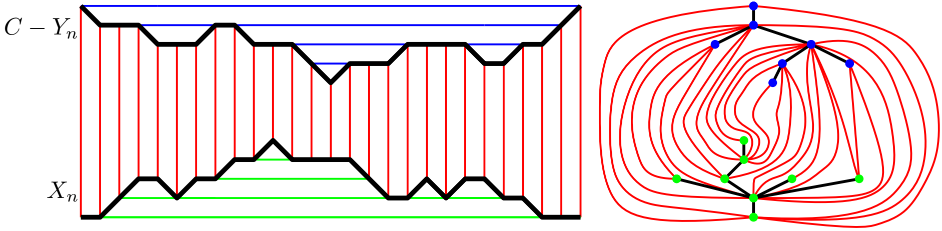
GFF and square subdivision for LQG measure

```
K = 8; fieldmultiplier = 1.5; squarefraction = .001;
phi=Re[Fourier[Table[(InverseErf[2 Random[]-1]+I InverseErf[2 Random[]-1])*If[j+k == 2,0,
     1/Sqrt[(Sin[(j-1)*Pi/2^K]^2+Sin[(k-1)*Pi/2^K]^2)]],{j,2^K},{k,2^K}]]];
MGFF=Exp[fieldmultiplier phi];C0 = squarefraction Sum[MGFF[[i,j]], {i,1,2^K}, {j,1,2^K}];
{ListPlot3D[phi],Graphics[Table[Table[If[Sum[MGFF[[2^k m+i,2^k n+j]],{i,1,2^k},{j,1,2^k}]<C0,
    {Hue[k/8],EdgeForm[Thin],Rectangle[{2^k m, 2^k n},{2^k m+2^k,2^k n+2^k}]}],
  {m,0,2^(K-k)-1},{n,0,2^(K-k)-1}], {k,0,K-1}]]}
```





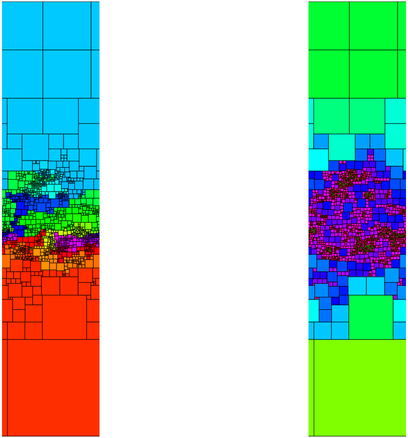
Recall Mullin bijection



When we delete the trees, we have a quadrangulation in which the edges come with a natural ordering. Also works for variant where tree root and dual-tree root are non-adjacent. Let's try a Smith embedding (with root and dual root for top and bottom) and color the squares according to that ordering.

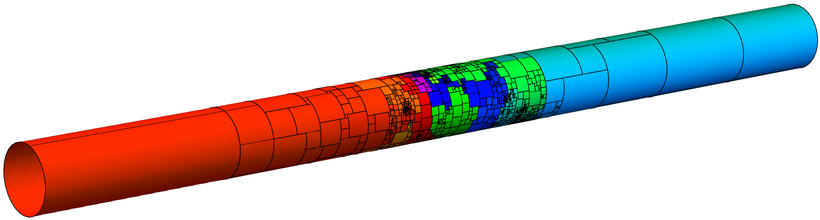
Smith embedding

```
M=M+Transpose[M];deg=Table[Sum[M[[i,j]],{i,1,n/2+2}],{j,1,n/2+2}];L=Table[If[i==j,-deg[[i]],M[[i,j]]],{i,1,n/2+2},{j,1,n/ 2+2}];
a=vertnumX[[minxloc]];b=vertnumY[[minyloc]]; For[j=1,j<=n/2+2,++j,L[[ a, j]]=0;L[[b, j]]=0];L[[a, a]] = 1; L[[b, b]] = 1;
v = Table[0, {j, 1, n/2 + 2}];  v[[a]]=1;w=LinearSolve[N[L], N[v]]; horiz =Table[0,{j,1,n+1}];
vertgap=Table[w[[vertnumY[[j]]]]-w[[vertnumX[[j]]]], {j,1,n+1}];horiz[[1]] = 0; For[j = 1, j <= n, ++j, horiz[[j+1]]=horiz[[j]]+
vertgap[[j]]]; horizgap=Abs[horiz[[n+1]]];g=Table[0, {j, 1, n}];count=1;sq[bot_, top_, left_, hue_]={Hue[hue], EdgeForm[Thin],
Rectangle[{left, bot}, {left + (top - bot), top}]};ssq[bot_,top_,left_,hue_]={Hue[-Log[Abs[top-bot]+.0000001]/6], EdgeForm[Thin],
Rectangle[{left, bot}, {left + (top - bot), top}]};g1=Table[sq[w[[vertnumX[[j]]]], w[[vertnumY[[j]]]], horiz[[j]], j/n],{j,1,n}];
g2=Table[ssq[w[[vertnumX[[j]]]], w[[vertnumY[[j]]]], horiz[[j]], j/n],{j,1,n}];
{Graphics[{g1,Translate[g1,{horizgap, 0}],Translate[g1,{2 horizgap, 0}]}, PlotRange->{{0, horizgap},{0,1}}],
 Graphics[{g2,Translate[g2,{horizgap, 0}],Translate[g2,{2 horizgap, 0}]}, PlotRange->{{0, horizgap},{0,1}}]}
```



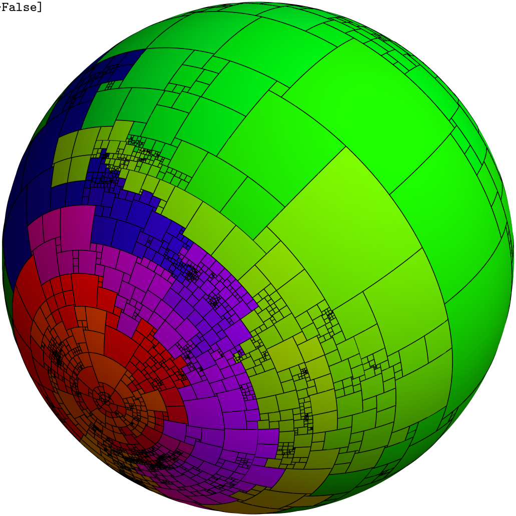
Cylinder picture of Smith embedding

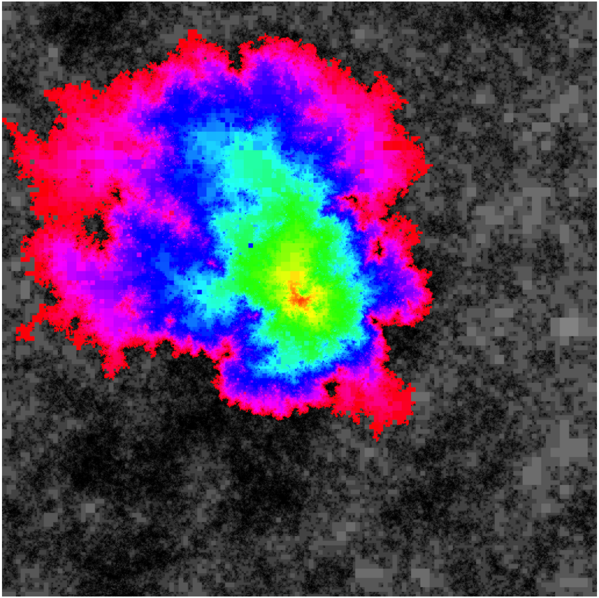
```
cutoff=.0001;rvsq[bot_, top_, left_, hue_]:= RevolutionPlot3D[{1, \[Theta]}, {\[Theta], (2 Pi/horizgap)bot,(2 Pi/horizgap)top
+.0000001},{p,(2 Pi/horizgap)left,(2 Pi/horizgap)(left+(top-bot)+.0000001)},Mesh->None,PlotStyle ->Hue[hue],BoundaryStyle ->
{None,Black}]; count=0; r=Table[0,{j,1,n}]; For[j=1, j<= n,++j,If[Abs[w[[vertnumX[[j]]]] - w[[vertnumY[[j]]]]] > .0001,
r[[++count]]=rvsq[w[[vertnumX[[j]]]],w[[vertnumY[[j]]]], horiz[[j]], j/n]]]; Show[Table[r[[j]], {j, 1, count}],
PlotRange ->All, Boxed ->False, Axes->False]
```



Projection onto the sphere

```
cutoff=.0001;spsq[bot_,top_,left_,hue_]:=SphericalPlot3D[1,{p,2ArcTan[Exp[(2 Pi/horizgap) (bot-1/2)]],2ArcTan[Exp[(2 Pi/horizgap)
 (top-1/2)]+ .0000001]},{\[Theta], (2 Pi/horizgap) left, (2 Pi/ horizgap) (left + (top - bot))+.0000001},Mesh->None,PlotStyle ->
 Hue[hue], BoundaryStyle -> {None, Black}];count = 0; For[j = 1, j<= n,++j, If[Abs[w[[vertnumX[[j]]]] - w[[vertnumY[[j]]]]] >
 .0001, r[[++count]]=spsq[w[[vertnumX[[j]]]], w[[vertnumY[[j]]]], horiz[[j]], j/n]]];Show[Table[r[[j]], {j,1,count}], PlotRange ->
  All, Boxed ->False,Axes ->False]
```



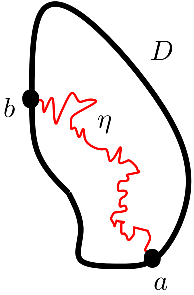


Metric growth on $\sqrt{8/3}$-LQG surface. Picture by Jason Miller.

$2 \leftrightarrow 3$. SLE-DECORATED LQG SPHERE IS EQUIVALENT TO PEANOSPHERE

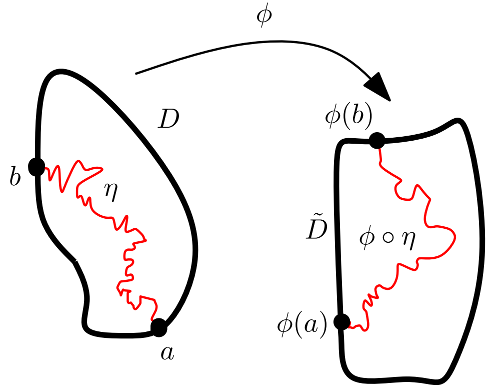
Random non-self-crossing path

Given a simply connected planar domain D with boundary points a and b and a parameter $\kappa \in [0, \infty)$, the **Schramm-Loewner evolution** SLE_κ is a random non-self-crossing path in $\overline{D}$ from a to b.



The parameter κ roughly indicates how "windy" the path is. Would like to argue that SLE is in some sense the "canonical" random non-self-crossing path. What symmetries characterize SLE?

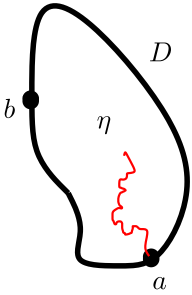
Conformal Markov property of SLE



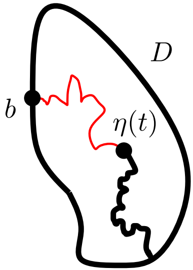
If ϕ conformally maps D to $\tilde{D}$ and η is an SLE_κ from a to b in D, then $\phi \circ \eta$ is an SLE_κ from $\phi(a)$ to $\phi(b)$ in $\tilde{D}$.

Markov Property

Given η up to a stopping time t...



law of remainder is SLE in $D \setminus \eta[0,t]$ from $\eta(t)$ to b.



Chordal Schramm-Loewner evolution (SLE)

THEOREM [Oded Schramm]: Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.

Chordal Schramm-Loewner evolution (SLE)

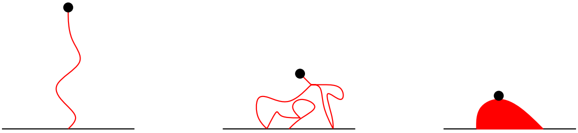
THEOREM [Oded Schramm]: Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.

Explicit construction: An SLE path γ from 0 to ∞ in the complex upper half plane H can be defined in an interesting way: given path γ one can construct conformal maps $g_t : \mathrm{H} \setminus \gamma([0,t]) \to \mathrm{H}$ (normalized to look like identity near infinity, i.e., $\lim_{z\to\infty} g_t(z) - z = 0$). In SLE_κ, one defines g_t via an ODE (which makes sense for each fixed z):

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

where $W_t = \sqrt{\kappa} B_t =_{LAW} B_{\kappa t}$ and B_t is ordinary Brownian motion.

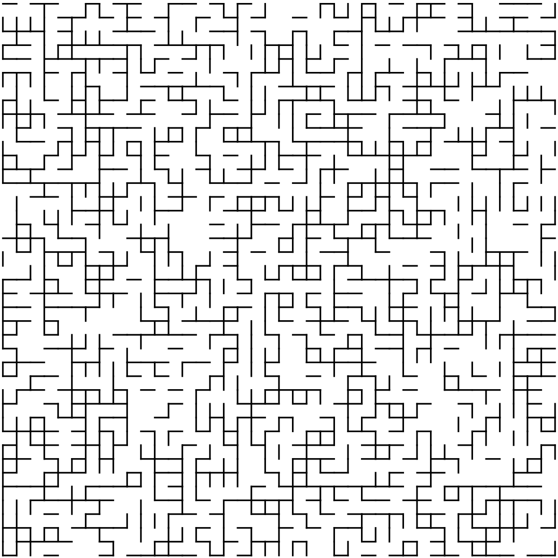
SLE phases [Rohde, Schramm]



$\kappa \le 4$ $\kappa \in (4, 8)$ $\kappa \ge 8$

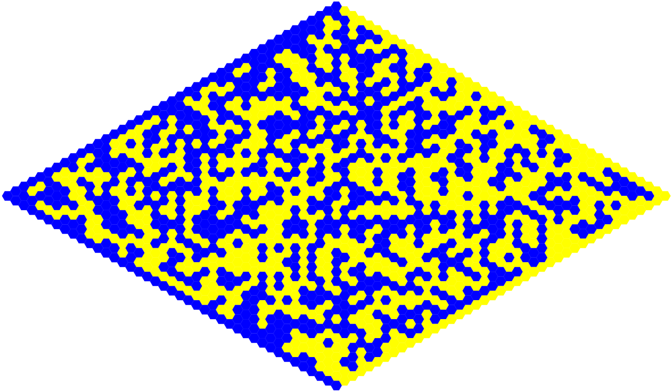
Bond percolation: toss coin for each edge

```
n=40;Graphics[{Table[If[RandomInteger[1]==1,Line[{{i,j},{i,j+1}}]],{i,0,n},{j,0,n-1}],
  Table[If[RandomInteger[1]==1,Line[{{i,j},{i+1,j}}]],{i,0,n-1},{j,0,n}]}]
```



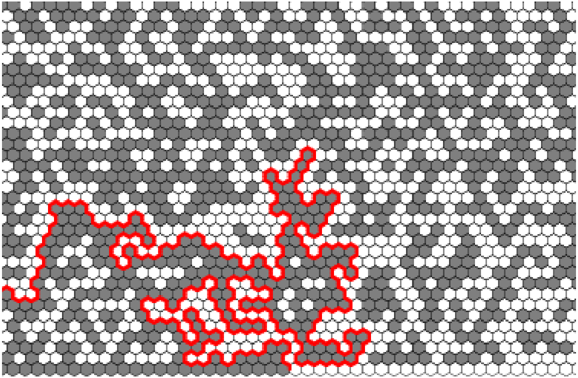
Site percolation: toss coin to color each face

```
n=40; Graphics[Table[{If[(i-n)(j-n)==0,Blue, If[i j==0,Yellow,If[RandomInteger[1]==1,Yellow,Blue]]],
 RegularPolygon[i{-Sqrt[3],-1}+j{-Sqrt[3],1},{1,0},6]},{i,0,n},{j,0,n}]]
```

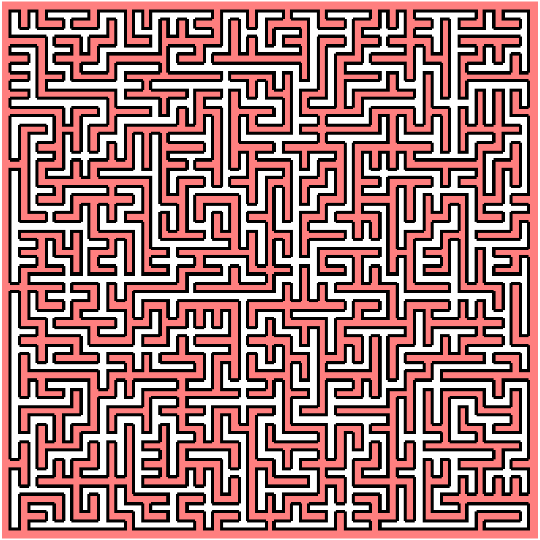


Left boundary: blue. **Right boundary:** yellow. **Blue-yellow interface:** loops plus one long path. Path converges in law to SLE_6. Stanislav Smirnov (ICM 2010 Fields Medal). Camia and Newman. **Ising model:** another random coloring with conformal invariant limit. SLE_3 and $\mathrm{SLE}_{16/3}$. Smirnov plus Chelkak, Duminil-Copin, Hongler, Izyurov, Kemppainen.

Percolation interface

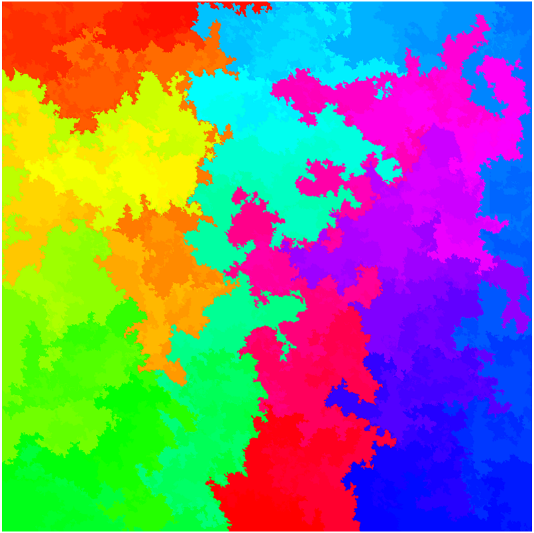


Uniform spanning tree (white), dual (red), interface (black)



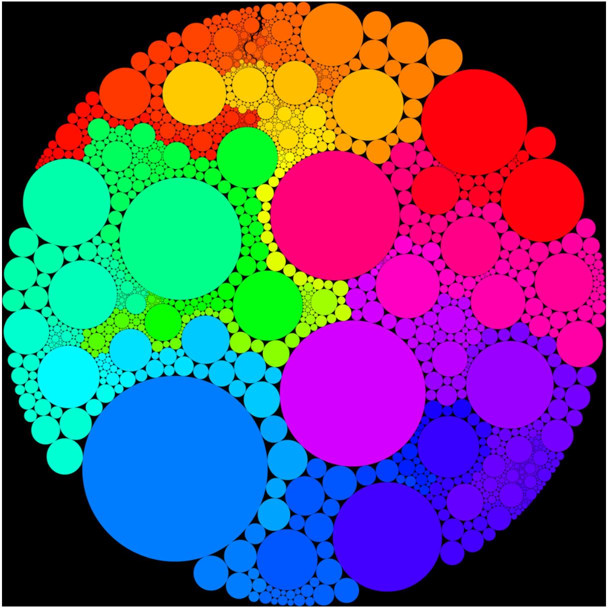
Black interface converges to SLE_8 loop. Lawler, Schramm, Werner.

Continuum space-filling SLE path



Picture by Jason Miller.

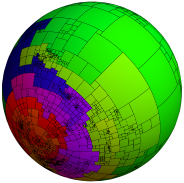
Similar construction with circle packings, also related to conformal maps.



Picture by Jason Miller, packed with Ken Stephenson's CirclePack.

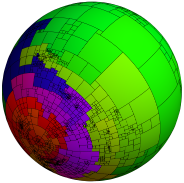
4. DEFINING THE MULTIPOINT FUNCTIONS OF CONFORMAL FIELD THEORY

Polyakov measure



STUDENT: How many ways to conformally embed a surface into the sphere?

Polyakov measure



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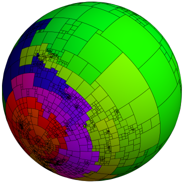
Polyakov measure



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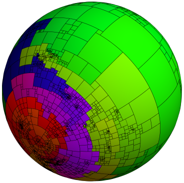

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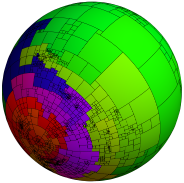
Polyakov measure



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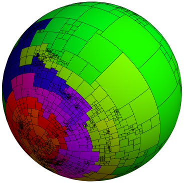

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INSTRUCTOR: Just integrate F w.r.t. the infinite measure.

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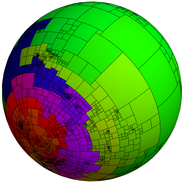

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STUDENT: How do you define expectation of F w.r.t. an infinite measure?
INSTRUCTOR: Just integrate F w.r.t. the infinite measure.
STUDENT: Got it. Can you show me what all these embeddings look like?

Liouville conformal field theory

STUDENT: So suppose F is "the amount of surface area parameterized by A" where A is a fixed ball (the Arctic Circle say). What would $\langle F \rangle$ be?

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INSTRUCTOR: Now you're talking. Yes, there are "relatively many" embeddings that assign a macroscopic mass to one or two of those countries, but "relatively few" assigning macroscopic mass to *all three*. So the expected "product of areas" will be finite in this case. More generally take three or more disjoint A_i. Consider the *product* of their areas: $\Big\langle \prod_i \int_{A_i} e^{\alpha_i \phi_i(x_i)} dx_i \Big\rangle$ where $\alpha_i = \gamma$. You can pull the integral outside the expectation and write this as $\int_{\prod A_i} \langle \prod e^{\alpha_i \phi(x_i)} \rangle \prod dx_i$. Integral of "multipoint function" $\langle \prod e^{\alpha_i \phi(x_i)} \rangle$.
STUDENT: What if I want a product of areas of balls, lengths of curves, fractal measures of fractal sets?...

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STUDENT: What if I want a product of areas of balls, lengths of curves, fractal measures of fractal sets?...
INSTRUCTOR: Use similar multipoint functions but let α_i be different.

Liouville conformal field theory

STUDENT: Are these multipoint functions easy to compute?

Liouville conformal field theory

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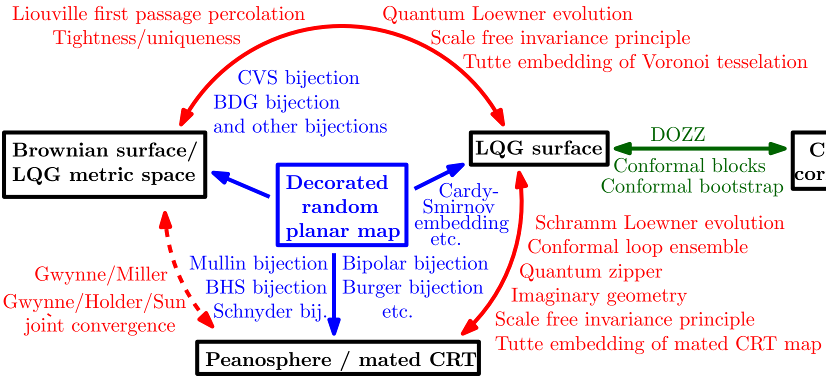
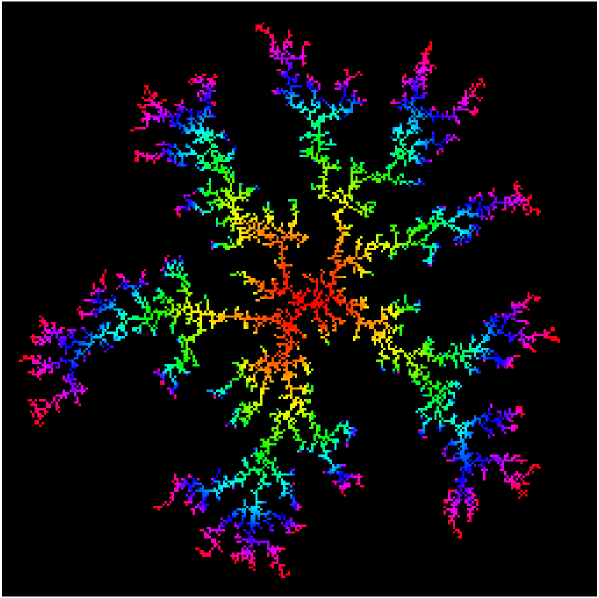
INSTRUCTOR: Ha! If ϕ were just a GFF then making formal sense of $\left\langle \prod e^{\alpha_i \phi(x_i)} \right\rangle$ would be easy. But once we fix the surface area to be one (or weight by its exponential) we get a difficult non-Gaussian integral. This problem inspired a whole subject called **conformal field theory** and its solution uses lots of amazing work (Belavin, Polyakov, Zamolodchikov brothers, David, Dorn, Teischner, Kupiainen, Guillarmou, Rhodes, Vargas, etc.) *Huge* subject with myriad ties to physics—quantum field theory, string theory, 2D statistical physics, etc. See Vargas in Quanta video `https://youtu.be/9uASADiYe_8?t=440`.

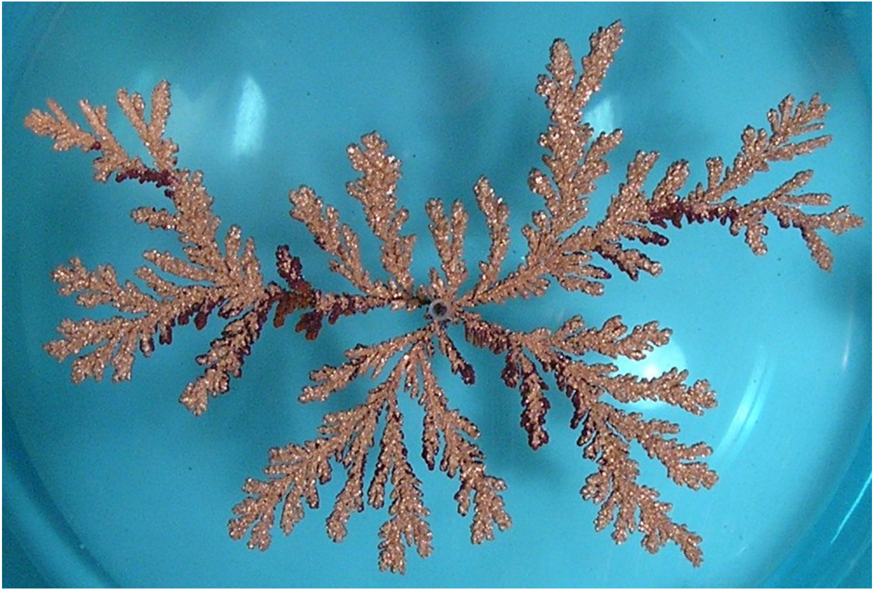
Connections and keywords



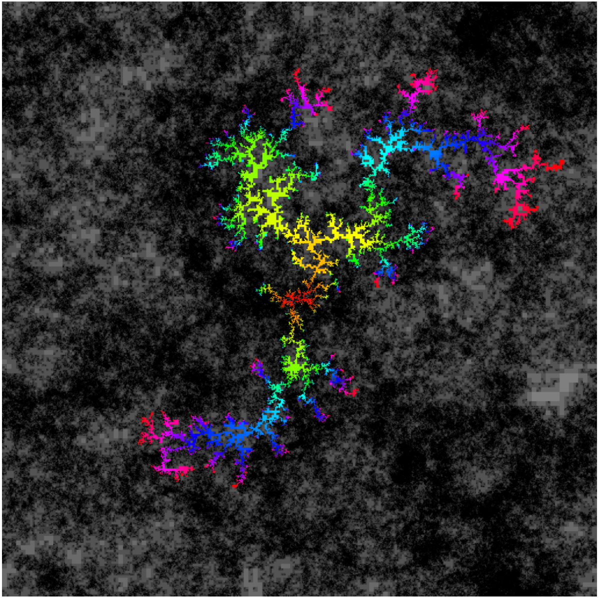
Thanks to co-authors and students: Tom Alberts, Morris Ang, Nathanaël Berestycki, Manan Bhatia, Bertrand Duplantier, Ewain Gwynne, Nina Holden, Richard Kenyon, Sungwook Kim, Greg Lawler, Asad Lodhia, Oren Louidor, Jason Miller, Andrei Okounkov, Minjae Park, Yuval Peres, Joshua Pfeffer, Rémi Rhodes, Oded Schramm, Nike Sun, Xin Sun, Vincent Vargas, Sam Watson, Menglu Wang, Wendelin Werner, David Wilson, Catherine Wolfram, Hao Wu and Pu Yu.



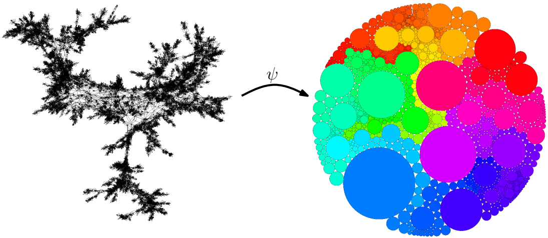
Some 2D models remain mysterious: Diffusion Limited Aggregation (DLA). Witten-Sander 1981.



DLA in nature: "A DLA cluster grown from a copper sulfate solution in an electrodeposition cell" (from Wikipedia)



DLA on a $\sqrt{2}$-LQG (picture by Jason Miller) is suprisingly more tractable.



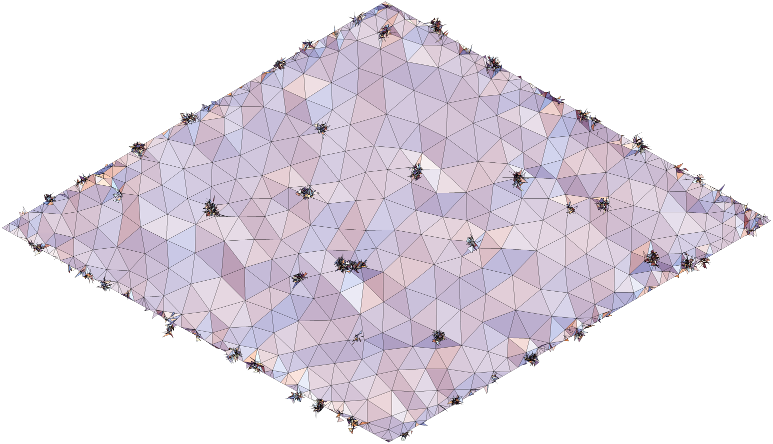
THANKS to those who helped improve the ICM notes: Tom Alberts, Morris Ang, Nathanael Berestycki, Olivier Bernardi, Sky Cao, Nicolas Curien, Bertrand Duplantier, Ewain Gwynne, Jean-François Le Gall, Grégory Miermont, Jason Miller, Ron Nissim, Minjae Park, Guillaume Remy, Rémi Rhodes, Steffen Rohde, Stanislav Smirnov, Yilin Wang and Wendelin Werner.

THANKS to the organizers.

THANKS to many anonymous referees.

And thank you for listening!

BONUS SLIDE: Exponential crochet by Tonya Khovanova. Amount of yarn needed grows like exponential of diameter d. For random planar map (approximatiing Brownian surface) yarn needed grows like d^4. Either way growth exceeds d^3 so there will be lots of compressing or stretching when d is large. This explains why it is hard to construct "nice" 3D embeddings of random triangulations when the number of triangles is too large.



BONUS SLIDE: Finite-area surfaces embedded in dimension 3 want to be “tree like.”' But if you start with a rhombic piece of triangular lattice, fix the boundary values, and let the rest of the surface evolve by Glauber dynamics, you start to get a minimal spanning surface decorated by “folded up trees” that dance around and merge. Related to Wilson loop expectations for Yang-Mills? Surfaces traced by Chatterjee’s string trajectories? See forthcoming work with Park, Pfeffer, Yu about Wilson loop expectations in 2D and flat surface sums/integrals.