How do you divide your (two dimensional) time?

SLE, **CLE**, the **GFF** and **Liouville quantum gravity** zippers/necklaces,

and also congratulations to Stas!

Scott Sheffield

MIT

ICM 2010, August 26

Outline

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Introduction: SLE and CLE

Gaussian free field

Liouville quantum gravity

Zippers and necklaces

Why study the Schramm-Loewner evolution?

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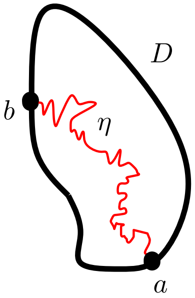
- It is a safe way to have fun with stochastic calculus (without causing a financial meltdown).
- It may help you naturally "divide time" into manageable pieces (particularly if you are a Liouville quantum gravity string, and by "time" you mean the intrinsic Riemannian surface parameterizing your trajectory).

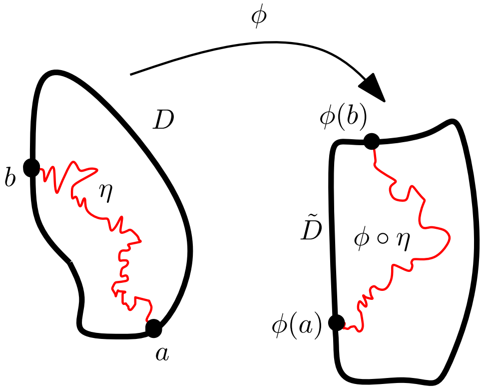
Schramm-Loewner evolution (SLE)

Given a simply connected planar domain D with boundary points a and b and a parameter $\kappa \in [0, \infty)$, the **Schramm-Loewner evolution** SLE_κ is a random path in $\overline{D}$ from a to b.



The parameter κ roughly indicates how "windy" the path is.

CONFORMAL INVARIANCE

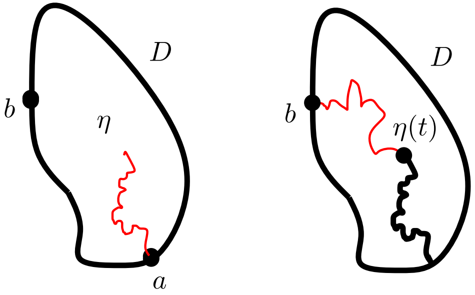


If ϕ conformally maps D to $\tilde{D}$ and η is an SLE$_\kappa$ from a to b in D, then $\phi \circ \eta$ is an SLE$_\kappa$ from $\phi(a)$ to $\phi(b)$ in $\tilde{D}$.

MARKOV PROPERTY

Given η up to a stopping time t...

law of remainder is SLE in $D \setminus \eta[0,t]$ from $\eta(t)$ to b.



Schramm-Loewner evolution (SLE)

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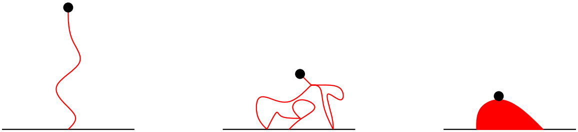
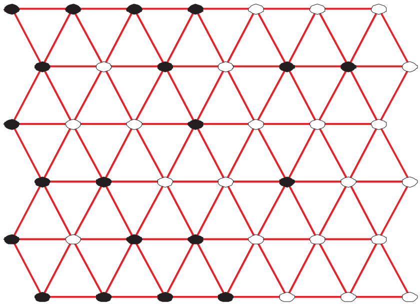
- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE.
- **VERY IMPORTANT:** by Riemann uniformization, SLE can be defined on *any* simply connected Riemannian surface with boundary, not just a planar domain. (The same will be true of *CLE*, to be defined later.)

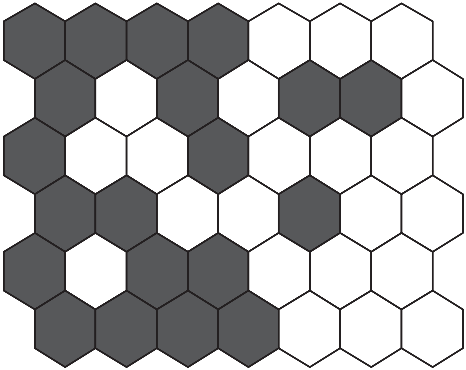
David Xianfeng Gu's conformal map images

SLE phases [Rohde, Schramm]



$\kappa \leq 4$ $\kappa \in (4, 8)$ $\kappa \geq 8$





What about other interfaces? The collection of loops?

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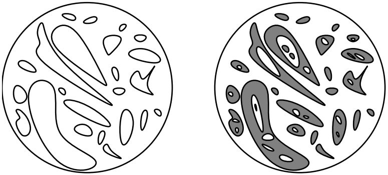

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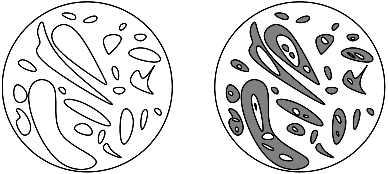
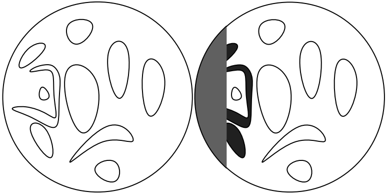


- **CONFORMAL INVARIANCE:** If ϕ conformally maps D to $\tilde{D}$ then the image of a CLE in D is a CLE in $\tilde{D}$.

- **MARKOV PROPERTY:** Fix a set A (with $D \setminus A$ simply connected). Given all the loops that hit A, the conditional law of the remaining loops is that of a CLE in the remaining domain.



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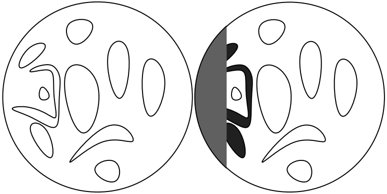

- **THEOREM [S, Wendelin Werner]:** Conformal invariance and the Markov property determine the law of a simple-loop CLE, up to the parameter $\kappa \in (8/3, 4]$.

CLE References

- *SLEs as boundaries of clusters of Brownian loops*, CRM [Werner]
- *Exploration trees and conformal loop ensembles*, Duke [S]
- *Conformal radii for conformal loop ensembles*, CMP [Schramm, S, Wilson]
- **Conformal loop ensembles: The Markovian characterization**, arXiv [S, Werner]
- **Conformal loop ensembles: Construction via loop-soups**, arXiv [S, Werner]

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- Philosophy: percolation and Ising models are the canonical simplest representatives of their "universality classes." Insights into these models are insights into the universe.
- Proofs: establish conformal invariance using new "holomorphic martingale observables" and notions of discrete analyticity.

Other SLE results

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- **Thm[Oded Schramm, S]:** Harmonic explorer, level sets of Gaussian free field scale to SLE_4.

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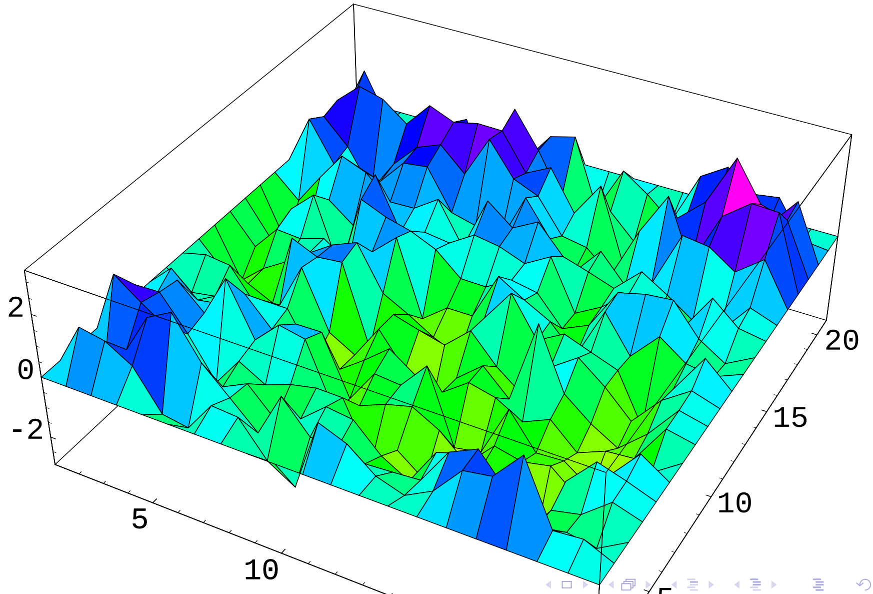
The discrete Gaussian free field

Let f and g be real functions defined on the vertices of a planar graph Λ. The **Dirichlet inner product** of f and g is given by

$$(f,g)_\nabla = \sum_{x\sim y} (f(x) - f(y))\,(g(x) - g(y))\,.$$

The value $H(f) = (f,f)_\nabla$ is called the **Dirichlet energy of f.** Fix a function f_0 on boundary vertices of Λ. The set of functions f that agree with f_0 is isomorphic to $\mathbb{R}^n$, where n is the number of interior vertices. The **discrete Gaussian free field** is a random element of this space with probability density proportional to $e^{-H(f)/2}$.

Discrete GFF on 20×20 grid, zero boundary



The continuum Gaussian free field

is a "standard Gaussian" on an *infinite* dimensional Hilbert space. Given a planar domain D, let $H(D)$ be the Hilbert space closure of the set of smooth, compactly supported functions on D under the conformally invariant *Dirichlet inner product*

$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dx dy.$$

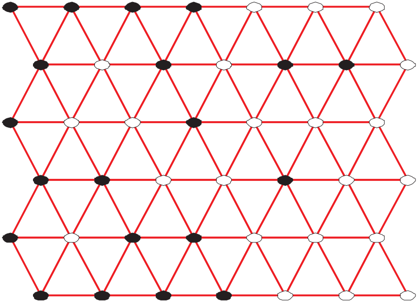
The GFF is the formal sum $h = \sum \alpha_i f_i$, where the f_i are an orthonormal basis for H and the α_i are i.i.d. Gaussians. The sum does not converge point-wise, but h can be defined as a *random distribution*—inner products (h, ϕ) are well defined whenever ϕ is sufficiently smooth.

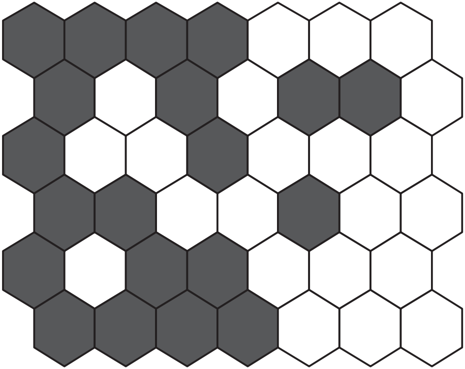
Some DGFF properties:

Zero boundary conditions: The Dirichlet form $(f, f)_\nabla$ is an inner product on the space of functions with zero boundary, and the DGFF is a standard Gaussian on this space.

Other boundary conditions: DGFF with boundary conditions f_0 is the same as DGFF with zero boundary conditions *plus* a deterministic function, which is the (discrete) harmonic interpolation of f_0 to Λ.

Markov property: **Given** the values of f on the boundary of a subgraph Λ' of Λ, the values of f on the remainder of Λ' have the law of a DGFF on Λ', with boundary condition given by the observed values of f on $\partial\Lambda'$.



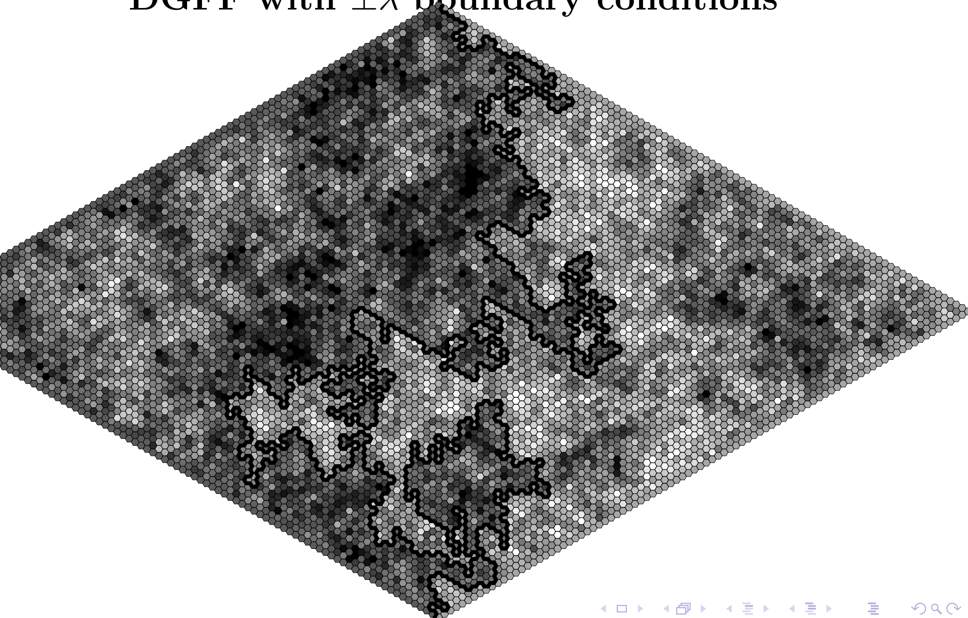


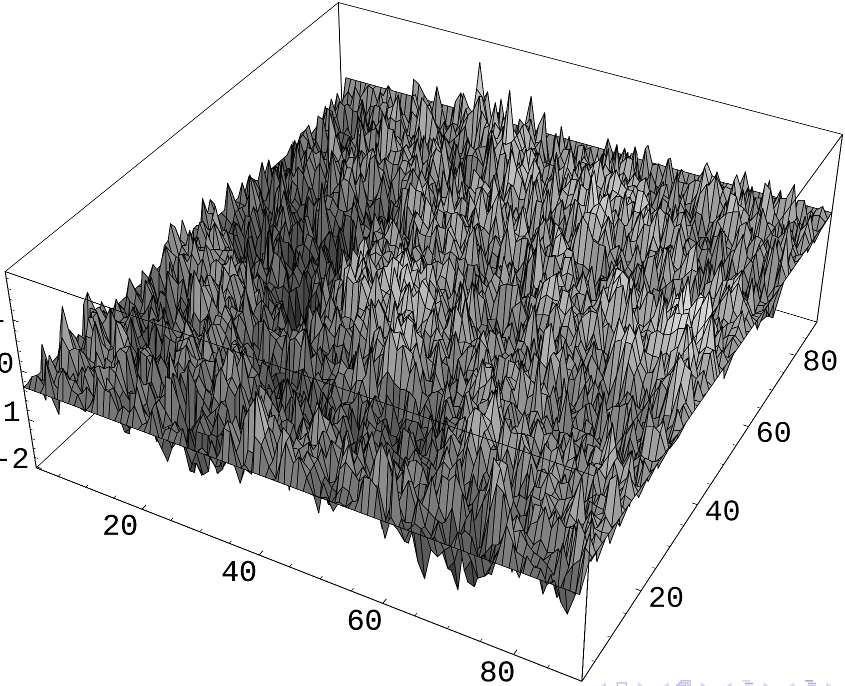
Scaling limit of zero-height contour line

Theorem (Schramm, S): If initial boundary heights are λ on one boundary arc and $-\lambda$ on the complementary arc, where λ is the constant $\sqrt{\frac{\pi}{8}}$, then the scaling limit of the zero-height interface (as the mesh size tends to zero) is SLE_4.

If the initial boundary heights are instead $-(1+a)\lambda$ and $(1+b)\lambda$, then as the mesh gets finer, the laws of the random paths described above converge to the law of $\mathrm{SLE}_{4,a,b}$.

DGFF with $\pm\lambda$ boundary conditions





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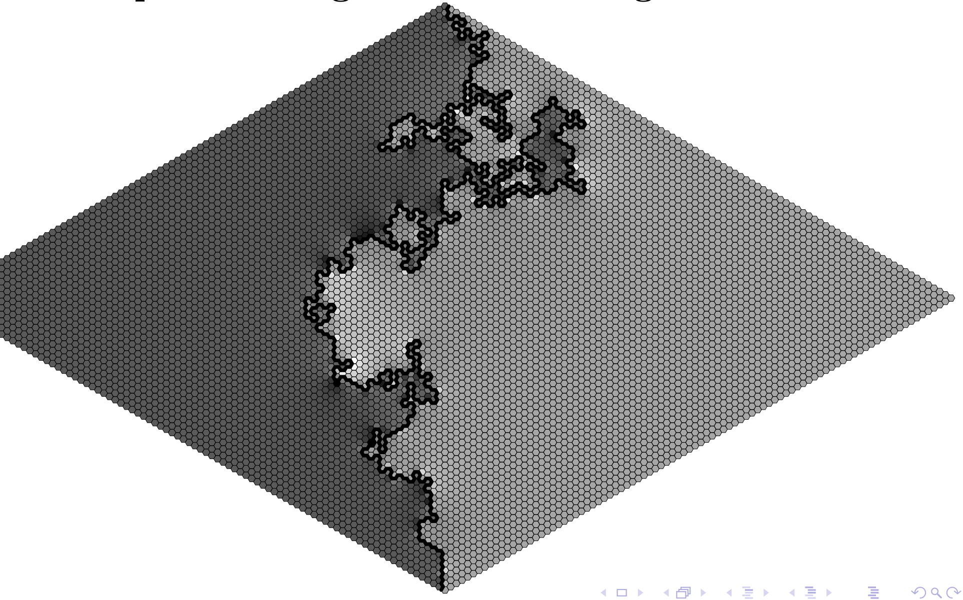
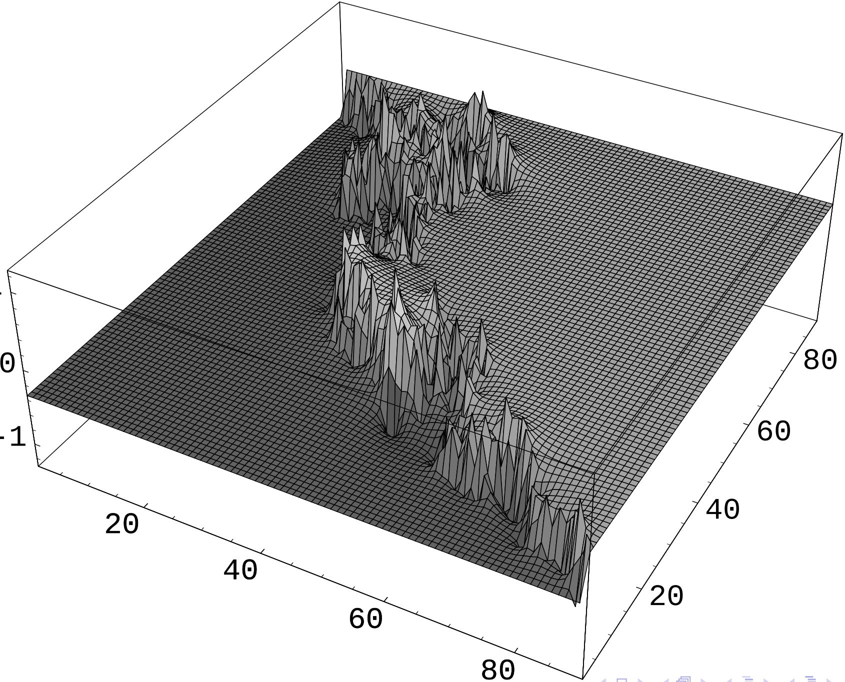
Expectations given values along interface





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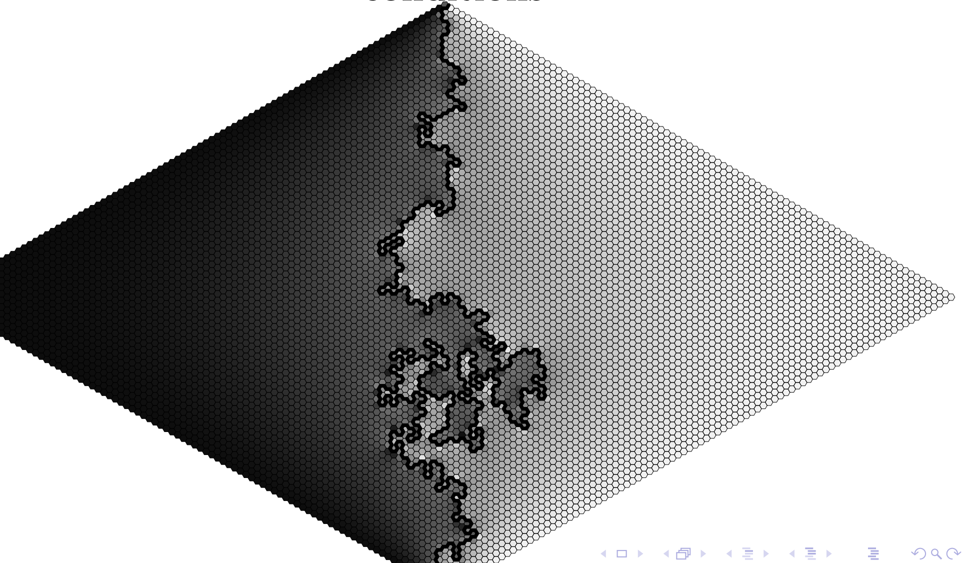
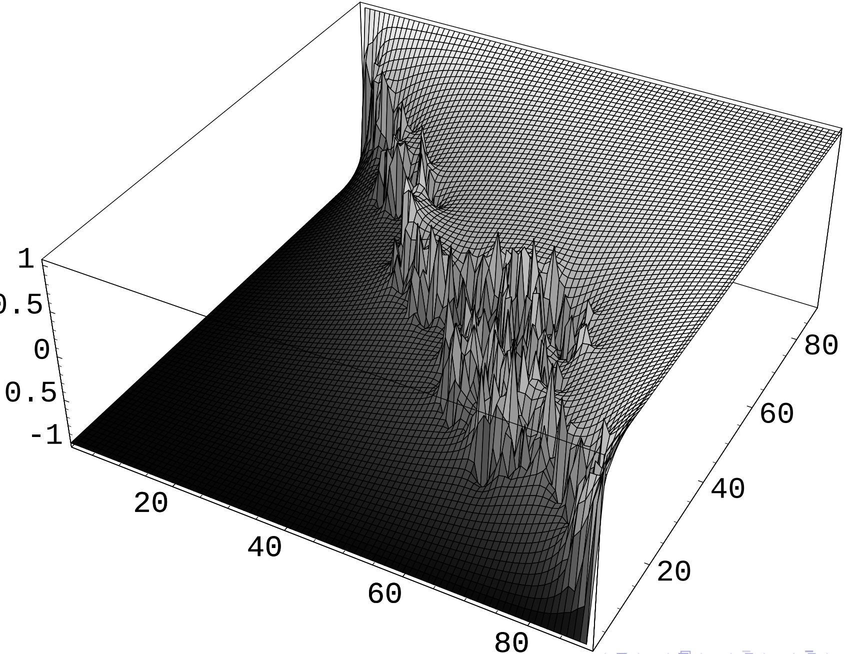
Expectations given interface, $\pm 3\lambda$ boundary conditions

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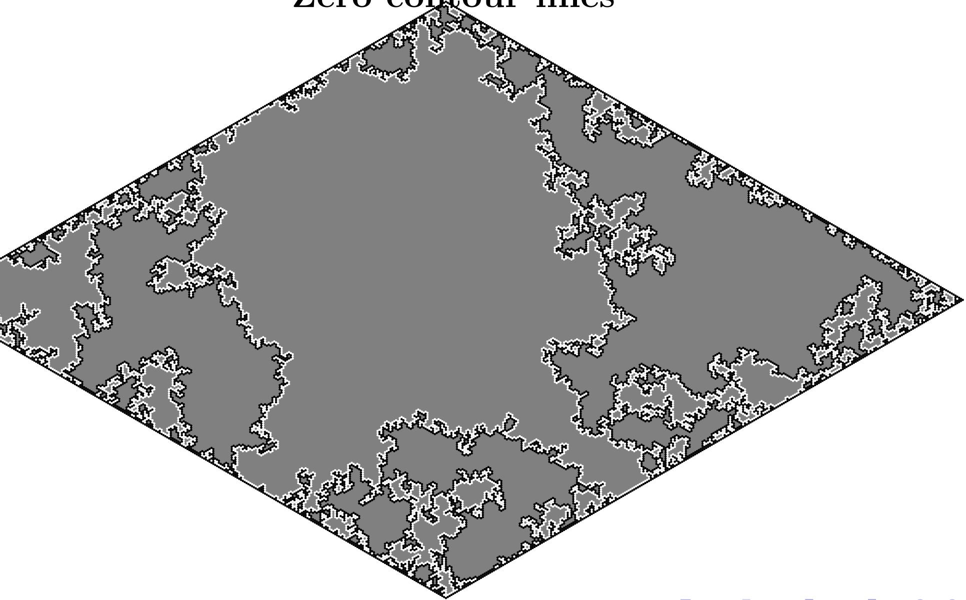
60
80

Zero contour lines



GFF References

- *The harmonic explorer and its convergence to SLE(4)*, Ann. Prob. [Schramm, S]
- *Local sets of the Gaussian free field, Parts I,II, and III*, Online lecture series: www.fields.utoronto.ca/audio/05-06 [S]
- **Contour lines of the two-dimensional discrete Gaussian free field**, Acta Math [Schramm, S]
- **A contour line of the continuum Gaussian free field**, arXiv [Schramm, S]

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"There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitude are given by the sums over all possible surfaces with fixed boundary."

A.M. Polyakov, Moscow 1981

How to construct a random 2D manifold?

- **Discrete approach**: Glue together unit squares or unit triangles in a random fashion. (Random quadrangulations, random triangulations, random planar maps, random matrix models.)

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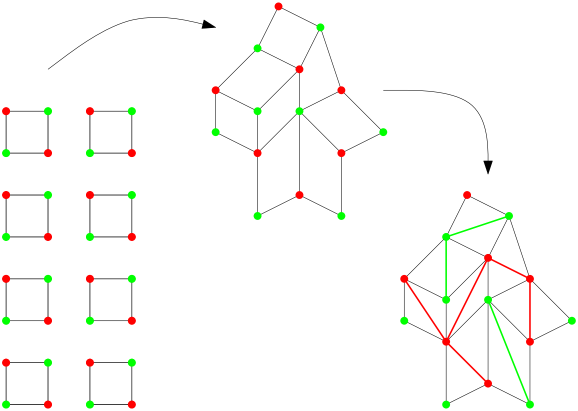
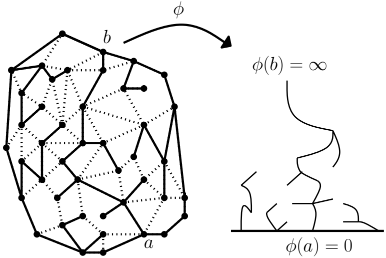
- **Discrete approach**: Glue together unit squares or unit triangles in a random fashion. (Random quadrangulations, random triangulations, random planar maps, random matrix models.)
- **Continuum approach**: Use conformal maps to reduce to a problem of constructing a random real-valued function on a planar domain or a sphere. Using the Gaussian free field for the random function yields (critical) Liouville quantum gravity.

Discrete construction: gluing squares



Discrete uniformizing maps



Planar map with one-chord-wired spanning tree (solid edges), plus image under conformal map to $\mathbb{H}$ (sketch).

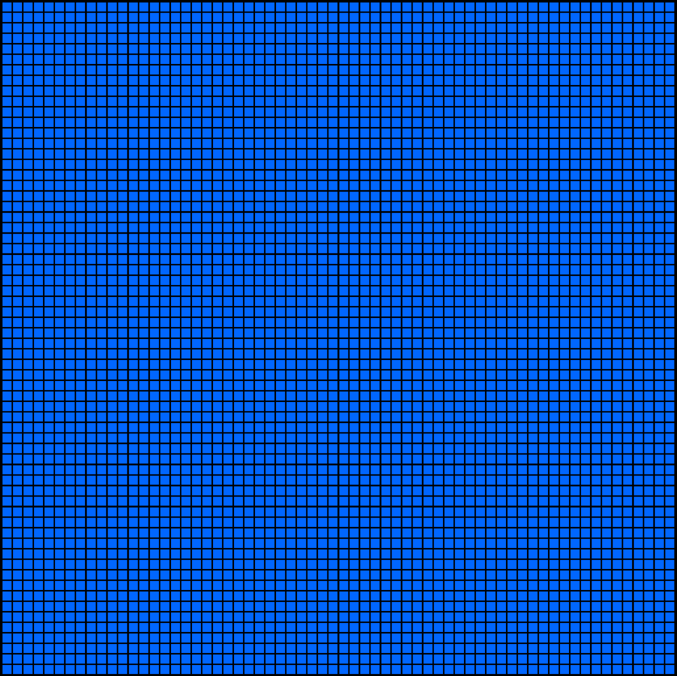
How about the continuum construction? Defining Liouville quantum gravity?

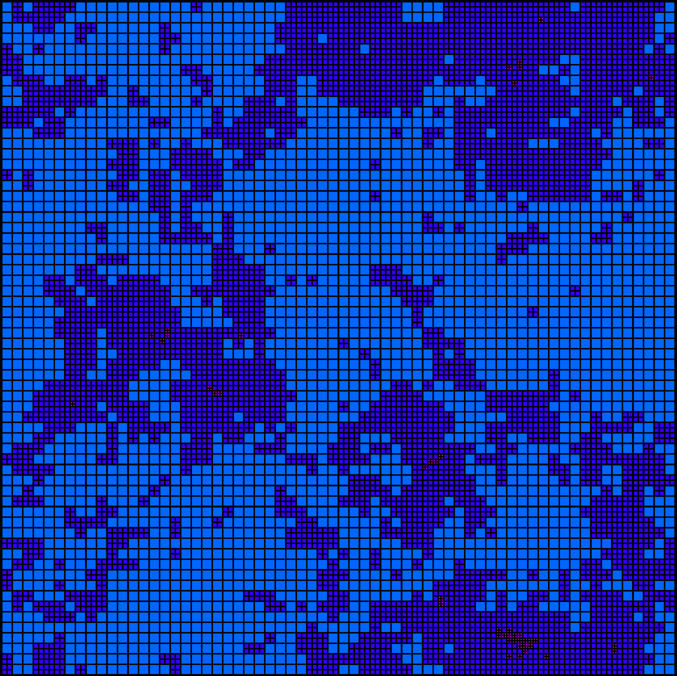
Constructing the random metric

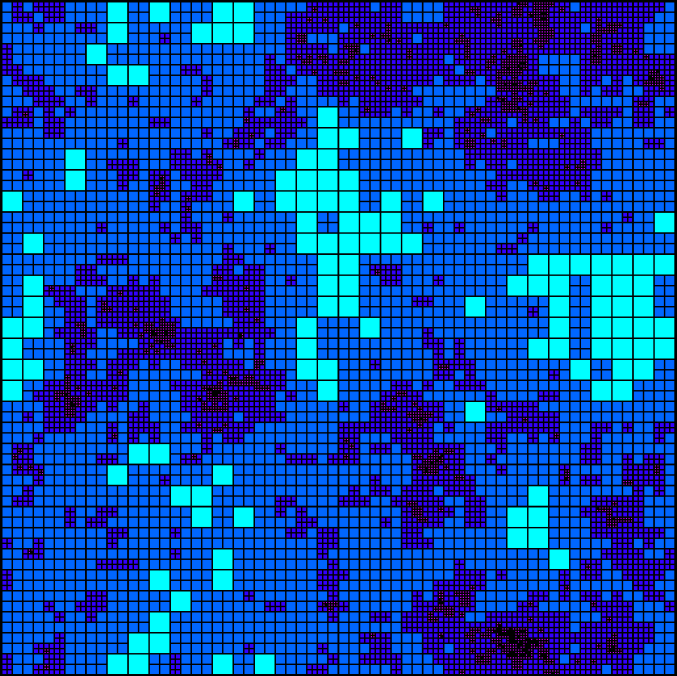
Let $h_\epsilon(z)$ denote the mean value of h on the circle of radius ϵ centered at z. This is almost surely a locally Hölder continuous function of (ϵ, z) on $(0,\infty) \times D$. For each fixed ϵ, consider the surface $\mathcal{M}_\epsilon$ parameterized by D with metric $e^{\gamma h_\epsilon(z)}(dx^2 + dy^2)$.

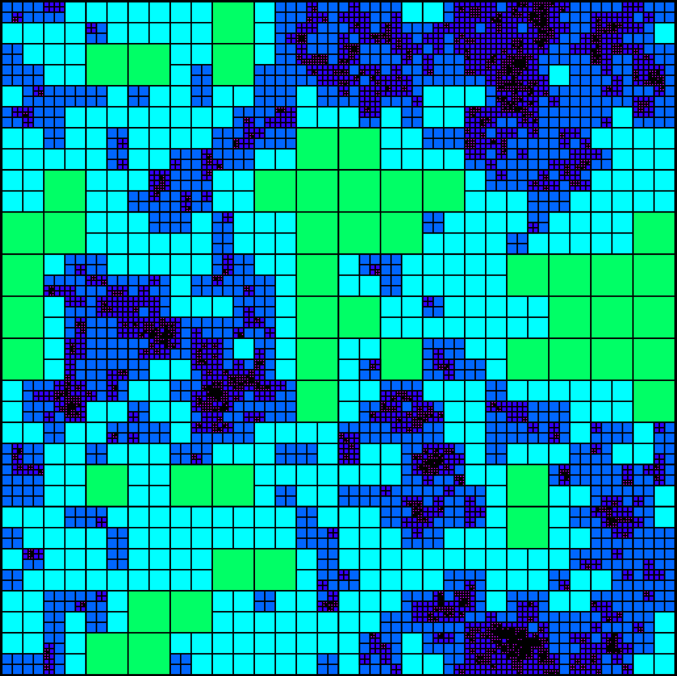
We define $\mathcal{M} = \lim_{\epsilon \to 0} \mathcal{M}_\epsilon$, but what does that mean?

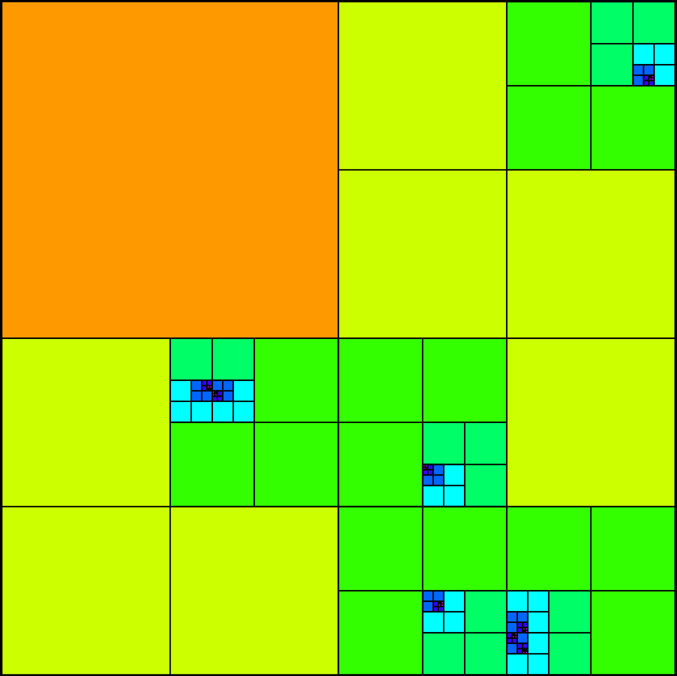
PROPOSITION: Fix $\gamma \in [0,2)$ and define h, D, and μ_ϵ as above. Then it is almost surely the case that as $\epsilon \to 0$ along powers of two, the measures $\mu_\epsilon := \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} dz$ converge weakly to a non-trivial limiting measure, which we denote by $\mu = \mu_h = e^{\gamma h(z)} dz$.











Knizhnik-Polyakov-Zamolodchikov (KPZ) Formula

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- Recently proved rigorously [Duplantier, S].

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- Similarly, the boundary length ν_h is almost surely the image under ψ of the measure $\nu_{\tilde{h}}$.

Defining *quantum surfaces*

- **DEFINITION:** A **quantum surface** is an equivalence class of pairs (D, h) under the equivalence transformations $(D, h) \to (\psi^{-1} D, h \circ \psi + Q \log |\psi'|) = (\tilde{D}, \tilde{h})$.

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- Area, boundary length, and conformal structure are well defined for such surfaces.

Liouville quantum gravity References

- **Liouville quantum gravity and KPZ**, arXiv [Duplantier, S]
- *Duality and KPZ in Liouville quantum gravity*, PRL [Duplantier, S]
- **Conformal weldings of random surfaces: SLE and the quantum gravity zipper**, Online draft: pims2010.web.officelive.com/Coursematerials.aspx [S]
- *Schramm-Loewner evolution and Liouville quantum gravity*, In preparation [Duplantier, S]

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Zipper/necklace stationarity: setup

- Fix $\kappa > 0$, $\kappa \neq 4$, and take $\gamma = \min\{\sqrt{\kappa}, \sqrt{16/\kappa}\}$. Let $h = \mathfrak{h}_0 + \tilde{h}$ where $\mathfrak{h}_0 = \frac{2}{\sqrt{\kappa}} \log(z)$ and $\tilde{h}$ is an instance of the free boundary GFF on the complex upper half plane $\mathbb{H}$.

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- Then $(\mathbb{H}, h)$ is a random quantum surface.

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- Fix $\kappa > 0$, $\kappa \neq 4$, and take $\gamma = \min\{\sqrt{\kappa}, \sqrt{16/\kappa}\}$. Let $h = \mathfrak{h}_0 + \tilde{h}$ where $\mathfrak{h}_0 = \frac{2}{\sqrt{\kappa}} \log(z)$ and $\tilde{h}$ is an instance of the free boundary GFF on the complex upper half plane $\mathbb{H}$.
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- Then $(\mathbb{H}, h)$ is a random quantum surface.
- Choose (independently of h) an SLE_κ path η from 0 to ∞ in $\mathbb{H}$.
- This produces a quantum surface with two special boundary points 0 and ∞ and a path connecting them.

Zipper/necklace stationarity: theorem

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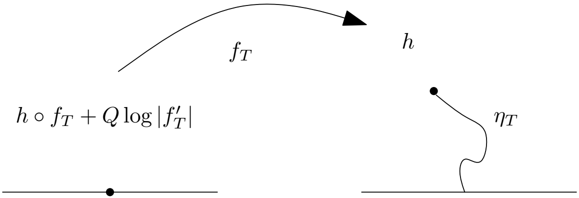
Zipper/necklace stationarity: theorem

- Take $(\mathbb{H}, h)$ and η as above and *cut* $\mathbb{H}$ along the path η up to some fixed time t.
- **THEOREM [S]:** The unbounded quantum surface that remains (after the cutting) has the same law as original law of $(\mathbb{H}, h)$.

Quantum gravity zipper



The map f_t "zips together" the positive and negative real axes, while f_t^{-1} "unzips" the path.

Conformal welding theorem

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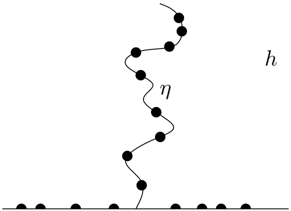
- In other words, the embedding of the two quantum surfaces (one on each side of γ) into $\mathbb{H}$ is a *conformal welding* of these surfaces.
- **Theorem[consequence of result of Jones-Smirnov]:** Homeomorphism between negative real axis and positive real axis (indentifying points of equal quantum length from 0) determines curve η obtained by zipping up.

Stationarity and matching quantum lengths



- Sketch of η with marks spaced at intervals of equal ν_h length.

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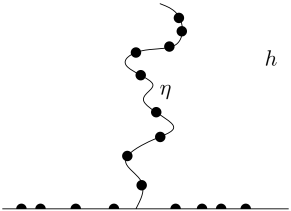

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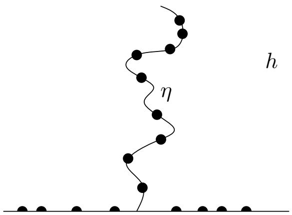
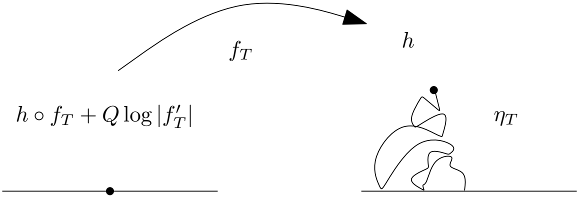


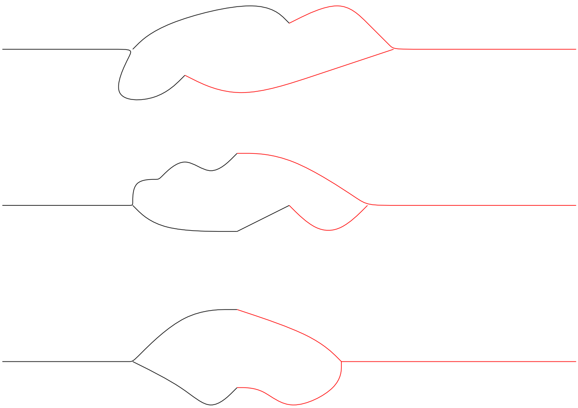
- Sketch of η with marks spaced at intervals of equal ν_h length.
- Semicircular dots on $\mathbb{R}$ are "zipped together" by f_t, then pulled apart (unzipped) by f_t^{-1}.
- The random pair (h, η) is stationary with respect to zipping up or down by a unit of (capacity) time.

Quantum gravity necklaces



Going from right to left, we lose a "chunk" of the quantum surface if $\kappa > 4$. Now if we take space-filling SLE and modify/reparameterize so that all successive chunks have unit quantum area...

Quantum gravity necklaces



then we can obtain a sequence of i.i.d. unit-area necklaces.

Sequences of necklaces

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- The LIFO inventory model admits a probabilistic analysis.

Scaling limits

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- The necklace decomposition theorems offer strong evidence that the discrete models converge to the continuum ones.
- They actually yield a proof that the convergence holds in particular topology (the "driving function topology").

Thanks to Oded Schramm (1961-2008), inventor of SLE,

and *thank you for coming!*