Poisson

18.600 Problem Set 5, due March 18

Welcome to your fifth 18.600 problem set! We'll be thinking more about Poisson random variables and the corresponding processes. Please stop by my weekly office hours (2-249, Wednesday 3 to 5) for discussion.

A. FROM TEXTBOOK CHAPTER FOUR:

1. Theoretical Exercise 16: Let X be a Poisson random variable with parameter λ. Show that $P\{X = i\}$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ. *Hint:* Consider $P\{X = i\}/P\{X = i - 1\}$.

2. Theoretical Exercise 25: Suppose that the number of events that occur in a specified time is a Poisson random variable with parameter λ. If each event is "counted" with probability p, independently of every other event, show that the number of events that are counted is a Poisson random variable with parameter λp. Also, give an intuitive argument as to why this should be so. As an application of the preceding result, suppose that the number of distinct uranium deposits in a given area is a Poisson random variable with parameter $\lambda = 10$. If, in a fixed period of time, each deposit is discovered independently with probability $\frac{1}{50}$, find the probability that (a) exactly 1, (b) at least 1, and (c) at most 1 deposit is discovered during that time.

B. FROM TEXTBOOK CHAPTER FIVE:

1. Problem 8: The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x} \quad x \geq 0$$

Compute the expected lifetime of such a tube.

2. Problem 11: A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than 1/4.

C. ANSWER THE FOLLOWING:

1. Compute the expectation of X^n where n is a positive integer and X is a uniform random variable on the interval $[0, 1]$.

2. How does the answer change if the random variable is instead taken to be uniform on $[0, L]$ for some constant L?

D. In Regular Bus City, there is a shuttle bus that goes between Stop A and Stop B, with no stops in between. The bus is perfectly punctual and arrives at Stop A at precise five minute intervals (6:00, 6:05, 6:10, 6:15, etc.) day and night, at which point it immediately picks up all passengers waiting. Citizens of Regular Bus City arrive at Stop A at Poisson random times, with an average of 5 passengers arriving every minute, and board the next bus that arrives.

(a) Suppose that you visit this city and that you arrive at Stop A at a time chosen uniformly at random from the times in a day. How long do you expect to have to wait until the next bus?

(b) How many citizens of Regular Bus City do you expect to be on the bus that you take?

In Poisson Bus City, there is a shuttle bus that goes between Stop A and Stop B, with no stops in between. The times at which the bus arrives at Stop A are a Poisson point process with one bus arriving every five minutes on average, day and night, at which point it immediately picks up all passengers waiting. Citizens of Poisson Bus City (like those of Regular Bus City) arrive at Stop A at Poisson random times, with an average of 5 passengers arriving every minute, and board the next bus that arrives.

(c) Suppose that you visit this city and that you arrive at Stop A at a time chosen uniformly at random from the times in a day. How long do you expect to have to wait until the next bus?

(d) How many citizens of Poisson Bus City do you expect to be on the bus that you take?

(e) Are the following two statements true or false? If they are both true, explain in words the apparent discrepancy:

   (i) When you visit, buses in Poisson Bus City seem on average to come twice as slowly and to be twice as crowded as those in Regular Bus City

   (ii) In both cities, buses come on average every five minutes and people come on average five times per minutes, so that over the long haul there are 25 people per bus on average—so buses are on average equally crowded in the two cities.

Remark: Poisson Bus City is not the worst case scenario. Suppose that buses come in pairs (one right behind the other) with the pairs arriving as a Poisson point process with one pair every 10 minutes on average. And suppose that whenever this happens, everybody gets in the first bus and leaves the second bus empty. Now if you arrive at a random time, you can expect your bus to take four times as long to come and be four times as crowded as in Regular Bus City (assuming that like others you get on the first bus in a pair). On a real life bus route with many stops, the closer a bus is to the bus ahead of it, the faster it can go (since it is picking up fewer passengers) which can lead to buses getting clumped together.

E. Each day (independently of all other days) Jill has a one in five thousand chance of hearing the basic details of the Peleponnesian War. She stores something in long term memory after hearing it 4 times. Use Poisson approximations to (approximately) answer the following:

(a) What is the probability that, by the time Jill is 10,000 days old, she has the Peleponnesian War in long term memory?

Alice is more of a reader than Jill and also has a better memory for trivia. Each day (independently of all other days) Alice has a one in one thousand chance of learning about the Peleponnesian War, and she stores the information in long term memory after hearing it 3 times.

(b) What is the probability that, by the time Alice is 10,000 days old, she has the Peleponnesian War in long term memory?

(c) If there are $10,000$ similar facts (each fact comes with same probabilities as above), how many of them do we expect that Jill knows but Alice doesn't (assuming that both are 10,000 days old)? Assume that for each given fact, the two Poisson random variables (number of times fact is heard by Alice and by Jill) are independent. (If the answer is small, then Jill should feel pretty lucky when one of these facts comes up while she is watching Jeopardy with Alice.)

F. In a place called Politically Balanced Country, there are $2N+1$ people (where N is an integer) voting in an election between Candidate A and Candidate B. All citizens are at least somewhat altruistic; they consider the expected value of good/prosperity/happiness. However, they use different subjective personal probability measures to calculate that expectation. Here is how that works. At some time a few weeks before the election, each citizen's brain mentally tosses a fair coin; with probabiliy 1/2 it becomes convinced that electing Candidate A will result in an extra expected \$1000 worth of good/prosperity/happiness for each citizen (compared to Candidate B). Otherwise it becomes convinced that electing Candidate B will result in an extra expected \$1000 worth of good/prosperity/happiness for each citizen (compared to Candidate A).

(a) Let Citizen X be one of the citizens. After tossing the mental coin and forming a personal probability measure, how much *overall* good does Citizen X expect (assuming this personal probability measure) to do by voting for the better candidate? (If Citizen X did not vote and the vote was tied, the race would be decided with a coin toss.) In other words, how much higher is the expected overall good (using Citizen X's probability measure) in the scenario in which Citizen X votes than in the scenario in which Citizen X doesn't vote.Use Stirling's approximation for $n!$ (look it up if you have to) to give an estimate for this value as a function of N.

(b) How much good does Citizen X expect to do *for herself personally* by voting (i.e., previous answer divided by $2N + 1$)?

Observe: if N is, say, 10^6 this value is huge! Citizen X therefore considers voting to be very important. Even if Citizen X is only slightly altruistic (willing to sacrifice one unit of personal happiness only if it brings about 10 units to others, say), and she considers voting a chore, she will decide that voting is worthwhile because of this large expected impact (even though the expected impact for herself personally is relatively small).

In Politically Less Balanced Country, the random process by which voters' opinions are formed is more complicated (the voters are not all independent of each other). As before, each candidate becomes convinced that one candiates is \$1000 better than the other in expectation, but the number who end up biased toward Candidate A is more or less uniformly distributed between $.9N$ and $1.1N$, so that there is a $1/(.2N) = 5/N$ probability that the vote (not counting Citizen X's vote) is tied.

(c) How much good does Citizen X expect (using her personal probability measure) to do by voting in this country?

Remark: When a vote is $N + 1$ to N, *all* voters in the majority are *pivotal* (i.e., they could have changed the election outcome by changing their vote). In Politically Balanced Country, an individual vote's chance to be pivotal (order $1/\sqrt{N}$, up to constant factor) makes its expected impact surprisingly high—much more than $1/(2N + 1)$ times the impact of the total election. This is also true in Politically Less Balanced Country to a lesser extent.

Remark: I'll let you consider whether *real* political disagreements are caused by differing probability measures or differing utility functions.