

Entropy, Martingales and Finance

18.600 Problem Set 10, due May 6

Welcome to your tenth 18.600 problem set! We'll be thinking a bit about the efficient market hypothesis, risk neutral probability, martingales, and the optional stopping theorem. These ideas are commonly applied to financial markets and prediction markets, but they come up in many other settings as well. Indeed, if X is any random variable with finite expectation, then as one observes more and more information, one's revised conditional expectation for X evolves as a martingale, and the optional stopping theorem applies to the sequence of revised expectations.

Suppose that you know you have an exactly $2^{-10} = 1/1024$ chance of dying during the next 12 months. (You can see at <https://www.ssa.gov/oact/STATS/table4c6.html> what fraction of US men and women your age die during a given year; the 1/1024 figure is way too high for women but actually a little low for men.) Now which would you prefer?

1. If you die, it happens a random time during the year (no knowledge in advance).
2. You witness the outcome of one coin toss every month over the course of ten months, and you die at the end of the year if all are heads.
3. All coins are tossed at once at the end of the year, and you die if they are all heads.

If you choose the second option, your conditional probability of dying that year will evolve as a martingale (starting at 1/1024, then jumping to 0 or 1/512, then jumping to 0 or 1/256, etc.) If you choose the third option, there is a single jump (from 1/1024 to either 1 or 0) that happens all at once. This choice posed here may seem morbid, but in fact real life poses frequent analogs of this question (involving cancer diagnoses, mammograms, etc.) and the answers are not easy. How much information about our revised chances do we really want to have? How do we respond emotionally to martingale ups and downs?

Any fan of action movies knows that the heroes frequently face circumstances where (based on non-movie logic) the conditional probability that they survive the adventure appears very low. (*C3PO: Sir, the possibility of successfully navigating an asteroid field is approximately 3,720 to 1.*) This (sequentially revised) conditional probability is like a martingale that gets very close to zero, then somehow comes back to a moderate value, then gets close to zero again, then returns to moderate, then gets *extremely close* to zero in a big climactic scene, and then somehow gets back to one. This behavior is unlikely for actual (non-movie) martingales.

This problem set also features problems about *entropy*, an extremely important concept in (for example) statistical physics, information theory, and data compression.

Please stop by my weekly office hours (2-249, Wednesday 3 to 5) for discussion.

A. FROM TEXTBOOK CHAPTER NINE:

1. Problem/Theoretical Exercise 13: Prove that if X can take on any of n possible values with respective probabilities P_1, \dots, P_n , then $H(X)$ is maximized when $P_i = 1/n, i = 1, \dots, n$. What is $H(X)$ equal to in this case?
2. Problem/Theoretical Exercise 15: A coin having probability $p = 2/3$ of coming up heads is flipped 6 times. Compute the entropy of the outcome of this experiment. (An “outcome” is a full toss sequence, e.g., $\{H, T, T, T, H, H\}$.)
3. Problem/Theoretical Exercise 17: Show that, for any discrete random variable X and function f ,

$$H(f(X)) \leq H(X).$$

B. Solve the following problems:

1. Suppose Harriet has 15 dollars. Her plan is to make one dollar bets on fair coin tosses until her wealth reaches either 0 or 50, and then to go home. What is the expected amount of money that Harriet will have when she goes home? What is the probability that she will have 50 when she goes home?
2. Harriet uses the phrase “I think A ” in a precise way. It means “the probability of the event A , conditioned on what I know now, is at least .5”. Which of the following statements is true:
 - (a) Harriet thinks she will lose.
 - (b) Harriet thinks that there will be a time when she thinks she will win.
 - (c) Harriet thinks that her amount of money will reach 20 and subsequently reach 10 before the game is over.
3. Consider a contract that at time N will be worth either 100 or 0. Let $S(n)$ be its price at time n for $0 \leq n \leq N$. If $S(n)$ is a martingale, and $S(0) = 47$, then what is the probability that the contract will be worth 100 at time N ?

C. Complete the derivation of the Black-Scholes formula for European call options, as outlined in the lecture slides, by explicitly computing

$$E[g(e^N)]e^{-rT},$$

where N is a normal random variable with mean $\mu = \log X_0 + (r - \sigma^2/2)T$ and $g(x) = \max\{0, x - K\}$.

D. Recall Pedro's story from lecture: there is a safe asset (whose value does not change) and a risky asset whose value goes up 15 percent with probability .53 and down 15 percent with probability .47 (independently of everything else) each month. But now suppose that Pedro has the option to divide his money between the risk-free and the risky assets.

1. If Pedro wishes to maximize the expected log of his wealth after one month, what fraction of his wealth should he invest in the risky asset?
2. If Pedro is allowed to rebalance his wealth between the two assets every month, what strategy should he use to maximize the expected log of the amount of money he will have after 100 years (1200 months)? (In particular, what fraction of his wealth should he invest in the risky asset each month? Should it be the same fraction every month?)
3. Suppose that Pedro uses this strategy for 100 years. Compute the mean and standard deviation of the log of Pedro's wealth at the end. Pedro's wealth itself after 100 years is a random variable. Estimate its median.

E. David Aldous of UC Berkeley devised and told me about the following problem. How many people in a given US presidential election cycle do we expect to have their probability of becoming president *at some point* exceed 10 percent? In other words, if we look at those prediction market charts (and pretend the plots are true continuous martingales) how many candidates do we *expect* will have their number at some point exceed 10 percent? Let's consider two ways to approach this problem.

- (a) Assume that at some point (well before the election) every person in the world has some small probability p to become president. The i th person has probability p_i and $\sum p_i = 1$. Assume that each person's probability evolves as a continuous martingale, and argue that each person has a $p_i/.1 = 10p_i$ chance to reach ten percent in the prediction markets at some time. Sum over all i to get an overall expected number of people who reach this threshold.
- (b) Imagine a gambler who adopts the following strategy. Whenever a candidate reaches 10 percent, the gambler buys a contract on that candidate for \$10 (which will pay \$100 if the candidate wins) and holds it until the end of the election. Then at the end of the election the gambler is certain to receive \$100 (since the gambler will have purchased a contract on the winner at the first time the winner's price reached \$10). Argue that by the optional stopping theorem, the gambler makes zero money in expectation. So the expected amount of money spent on contracts must be \$100.

Here is a variant of this story. Suppose there is a .85 chance you will get married eventually (at least once). Imagine that some aliens watching your life from afar are

placing bets on who your first spouse will be, and that the contract prices evolve as continuous martingales. Call somebody an “almost first spouse” if at some point the market probability that this person is your first spouse exceeds .5.

- (c) Assuming continuity of martingales, how many almost first spouses do you expect to have over your lifetime?
- (d) Call a person a “serious contender” if their probability exceeds .1 at some point. How many serious contenders do you expect to have?

F. Imagine that at this particular moment on the currency market, one dollar has the same value as one euro. Let R be the event that (at some time during the next year) the euro rises in value relative to the dollar, so that the euro becomes worth two dollars. Let P_d be the cost, in dollars, of a contract that gives you one dollar if and when the event R occurs. Let P_e be the cost, in euros, of a contract that gives you one euro if and when event R occurs. Argue that one should expect $P_e = 2P_d$.

Remark: Assuming no interest, you can interpret P_d as the risk neutral probability of R (using dollars as the *numéraire*), but you can also interpret P_e as the risk neutral probability of R (using euros as the *numéraire*). This should convince you that the risk neutral probability of R cannot *always* be interpreted as a “general consensus of the subjective probability that R will occur”. (That interpretation is only reasonable if the value of money does not depend on whether R occurs.)

Congratulations on finishing (or at least reading to the end of) your final problem set! In addition to the abstract mathematics, your problem sets have included a range of stories about different subjects. My hope is that the stories have helped you integrate probability into the way you think about the world, and that you are ready to seriously contemplate Powerball, poker, tennis, xkcd, traffic, subprime lending, moon lassoing, portfolio management, bus regularity, the Peloponnesian War, bacteria, coin toss bias, heart attack treatment, political polarization, primary voting, mystery envelopes, Taylor Swift, IVF, the Kelly strategy, free throw contests, reproducibility, drug trials, cigarettes, celery, university admissions, Higgs bosons, stock options, hedge funds, weather, nutrition, Yelp reviews, and true love.

I also hope that some of you will use probability to solve big problems: to fundamentally improve the way our species approaches science, medicine, criminal justice, economics, engineering, and so forth. Failing that, I hope you’ll at least have fun with all of this.

Best of luck!