Permutations, Poker, and Powerball

18.600 Problem Set 1, due February 12

Welcome to your first 18.600 problem set! There will be ten problem sets this semester, each with a different theme, and each including a mix of problems of my own design and problems from the Sheldon Ross 8th edition textbook.

Part of the reason for including problems from the textbook is to remind you that the textbook exists, and to encourage you to read it (any of the 6th through 9th editions will do). You can study in various ways: reading the book, attending lecture, reviewing slides online, reviewing notes, working on problems alone, working on problems with friends, timing yourself on practice exams, attending office hours, and so forth. Based on my personal internal subjective probability measure... you're more likely to remember this material long term if you study in a variety of different ways. You'll probably figure out a few that work especially well for you.

This is a course in pure mathematics. Mathematical derivation. Mathematical truth. But students are encouraged to think about applications. What are we doing wrong in medicine? In traffic management? In college admissions? In teaching? In grading? In food preparation? Are there some simple pieces of advice that, if followed, would make us all better off? Are there other commonly accepted pieces of advice that we should all stop following? If so, how can we find out what these things are? These questions go well, well beyond the scope of this course, but it is hard to address them without thinking about probability.

The first problem set is about basic combinatorics. Cards, hats, seating arrangements, balls, binomial and multinomial coefficients. It will help to keep these stories in mind as the course progresses.

Please stop by my weekly office hours (2-249, Wednesday 3 to 5) for discussion. I have a large office with room for several people to work and talk in a relaxed way. I'm also usually free just after lecture.

A. FROM ROSS 8th EDITION CHAPTER ONE:

1. **Problem 10:** In how many ways can 8 people be seated in a row if

    (a) there are no restrictions on the seating arrangement?

    (b) persons A and B must sit next to each other?

    (c) there are 4 women and 4 men and no 2 men or 2 women can sit next to each other?

    (d) there are 5 men and they must sit next to each other?

    (e) there are 4 married couples and each couple must sit together?

2. **Problem 26:** Expand $(x_1 + 2x_2 + 3x_3)^4$.

3. **Theoretical Exercise 8:** Prove that
   $\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \ldots + \binom{n}{r}\binom{m}{0}$. *Hint:* Consider a group of n men and m women. How many groups of size r are possible?

4. **Theoretical Exercise 11:** The following identity is known as Fermat's combinatorial identity:

   $$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \quad n \geq k.$$

   Give a combinatorial argument (no computations are needed) to establish this identity. *Hint:* Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?

B. Suppose you have 12 (distinguishable) hats and 4 (distinguishable) people. How many ways are there to divide the 12 hats among the 4 people with each person getting exactly three hats?

C. Consider permutations $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

1. How many such σ have only one cycle, i.e., have the property that $\sigma(1), \sigma \circ \sigma(1), \sigma \circ \sigma \circ \sigma(1), \ldots$ cycles through all elements of $\{1, 2, \ldots, n\}$?

2. How many σ are fixed-point-free involutions, i.e., have the property that for each j, $\sigma(j) \neq j$ but $\sigma \circ \sigma(j) = j$?

D. In the US lottery game of Powerball one is required to choose an (unordered) collection of five numbers from the set $\{1, 2, \ldots, 69\}$ (the white balls) along with another number from the set $\{1, 2, \ldots, 26\}$ (the red ball). So there are $\binom{69}{5} \cdot 26 = 292201338$ possible Powerball outcomes. You make your selection (five white, one red), the Powerball people choose theirs randomly, and you win if there is a match. Suppose that you have already chosen your numbers (the unordered set of five white, and the one red). How many possible Powerball outcomes match *exactly one* of your five white numbers (regardless of whether they match the red number)? How many match exactly two of your five white numbers? How many match exactly three? How many match one red ball *plus* exactly two white balls?

Remark: People in the US spend over 70 billion per year on lottery tickets (about 32 percent of which is returned in big lottery payouts). That's well over two hundred dollars per person, with many players spending thousands of dollars a year. The psychological appeal may be hard for us to understand. But the fact that regular players get "close" now and then (matching two or three numbers) may be part of what keeps them coming back.

E. How many five card poker hands (from the standard deck with 52 cards; four suits and 13 face values) have five distinct face values?