18.600: Lecture 9

Expectations of discrete random variables

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Outline

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Defining expectation

Functions of random variables

Motivation

Expectation of a discrete random variable

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- Represents weighted average of possible values X can take, each value being weighted by its probability.

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- Answer: $\frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}6 = \frac{21}{6} = 3.5$.

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- State space is $\{(H,H),(H,T),(T,H),(T,T)\}$ and summing over state space gives $E[X] = \frac{1}{4}2 + \frac{1}{4}1 + \frac{1}{4}1 + \frac{1}{4}0 = 1$.

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- If the state space S is countable, is it possible that the sum $E[X] = \sum_{s \in S} P(\{s\})X(s)$ somehow depends on the order in which $s \in S$ are enumerated?
- In principle, yes... We only say expectation is defined when $\sum_{s \in S} P(\{x\})|X(s)| < \infty$, in which case it turns out that the sum does not depend on the order.

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- Generally, $E[aX + b] = aE[X] + b = a\mu + b$.

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- Alternatively, use symmetry. Expected number of heads should be same as expected number of tails.
- This implies $E[X] = E[n - X]$. Applying $E[aX + b] = aE[X] + b$ formula (with $a = -1$ and $b = n$), we obtain $E[X] = n - E[X]$ and conclude that $E[X] = n/2$.

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- Can extend to more variables $E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$.

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- Can write total number with own hat as $X = X_1 + X_2 + \ldots + X_n$.
- Linearity of expectation gives $E[X] = E[X_1] + E[X_2] + \ldots + E[X_n] = n \times 1/n = 1$.

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- **Comes up everywhere** probability is applied.

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- Let's assume $u(0) = 0$ and $u(1) = 1$. Then $u(x) = y$ means that you are indifferent between getting 1 dollar no matter what and getting x dollars with probability $1/y$.