

18.600: Lecture 7

Bayes' formula and independence

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Outline

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Independence

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Recall definition: conditional probability

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- ▶ Equivalent statement: $P(EF) = P(F)P(E|F)$.
- ▶ Call $P(E|F)$ the “conditional probability of E given F ” or “probability of E conditioned on F ”.

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- ▶ What is $P(D|T)$?

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- ▶ What if ratio is $1/P(A)$?

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- ▶ Repeat procedure as new evidence emerges.
- ▶ Caution required. My idea to check whether B occurred, or is a lawyer selecting the provable events B_1, B_2, B_3, \dots that maximize $P(A|B_1 B_2 B_3 \dots)$? Where did my probability estimates come from? What is my state space? What assumptions am I making?

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- ▶ Do we use Bayes subconsciously to update hunches?
- ▶ Should we think of Bayesian priors and updates as part of the epistemological foundation of science and statistics?

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- ▶ Say I think A is 5 times as likely as A^c , and $P(B|A) = 3P(B|A^c)$. Given B , I think A is 15 times as likely as A^c .

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- ▶ It $P(\cdot)$ is the *prior* probability measure and $P(\cdot|F)$ is the *posterior* measure (revised after discovering that F occurs).

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- ▶ Yes: probability of each event is $1/2$ and probability of both is $1/4$...
- ▶ despite intuitive sense that oddness of the number of heads “depends” on the first coin.

Independence of multiple events

- Say $E_1 \dots E_n$ are independent if for each $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$ we have $P(E_{i_1} E_{i_2} \dots E_{i_k}) = P(E_{i_1})P(E_{i_2}) \dots P(E_{i_k})$.

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- ▶ Does pairwise independence imply independence?
- ▶ No. Consider these three events: first coin heads, second coin heads, odd number heads. Pairwise independent, not independent.

Independence: another example

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- ▶ Generalize to $n > 7$ cards. What is $P(E_{1,7}|E_{1,2}E_{1,3}E_{1,4}E_{1,5}E_{1,6})$?