18.600: Lecture 5

Problems with all outcomes equally like, including a famous hat problem

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Outline

Equal likelihood

A few problems

Hat problem

A few more problems

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- What is $P(A)$ for a general set $A \subset S$?
- Answer: $|A|/|S|$, where $|A|$ is the number of elements in A.

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- In a room of 23 people, what is the probability that two of them have a birthday in common?

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Recall the inclusion-exclusion identity

- $$P(\cup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \ldots$$
$$+ (-1)^{(r+1)} \sum_{i_1<i_2<\ldots<i_r} P(E_{i_1}E_{i_2}\ldots E_{i_r})$$
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- The notation $\sum_{i_1 < i_2 < i_r}$ means a sum over all of the $\binom{n}{r}$ subsets of size r of the set $\{1, 2, \dots, n\}$.

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- $1 - P(\cup_{i=1}^n E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots \pm \frac{1}{n!} \approx 1/e \approx .36788$

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- What is the probability of a two-pair hand in poker?
- What is the probability of a bridge hand with 3 of one suit, 3 of one suit, 2 of one suit, 5 of another suit?