18.600: Lecture 36

Risk Neutral Probability and Black-Scholes

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Outline

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Black-Scholes

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- Brownian motion (as mathematically constructed by MIT professor Norbert Wiener) is a *continuous time martingale*.
- Black-Scholes theory assumes that the log of an asset price is a process called *Brownian motion with drift* with respect to *risk neutral probability*. Implies option price formula.

See desk at office hours

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- Risk neutral probability is the probability determined by the market betting odds.

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- **Risk neutral probability of event A:** $P_{RN}(A)$ denotes

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- Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment), P_{RN} satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events A_j are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$

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- **Arbitrage example:** if A and B are disjoint and $P_{RN}(A \cup B) < P(A) + P(B)$ then we sell contracts paying 1 if A occurs and 1 if B occurs, buy contract paying 1 if $A \cup B$ occurs, pocket difference.

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- Now, suppose there are only 2 outcomes: A is event that economy booms and everyone prospers and B is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think A has a .5 chance to occur, do we expect $P_{RN}(A) > .5$ or $P_{RN}(A) < .5$?

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- Answer: $P_{RN}(A) < .5$. People are risk averse. In second scenario they need the money more.

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- Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to team's prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of what well-informed informed experts would consider the true probability.

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- For simplicity, we focus on fixed time T, fixed interest rate r in this lecture.

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- Listener: Yeah, that's what I thought.

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- **Aside:** So-called **fundamental theorem of asset pricing** states that (assuming no arbitrage) interest-discounted asset prices are martingales with respect to risk neutral probability. Current price of stock being $E_{RN}(X)e^{-rT}$ follows from this.

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- **Conclusion:** If g is any function then the price of a contract that pays $g(X)$ at time T is

$$E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$$

where N is normal with mean μ and variance $T\sigma^2$.

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- Write this as

$$e^{-rT}E_{RN}[\max\{0, e^N - K\}] = e^{-rT}E_{RN}[(e^N - K)1_{N \geq \log K}]$$

$$= \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}}(e^x - K)dx.$$

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- Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function Φ.
- Price of European call is $\Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT}$ where $d_1 = \frac{\ln(\frac{X_0}{K})+(r+\frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln(\frac{X_0}{K})+(r-\frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$.

Determining risk neutral probability from call quotes

- If $C(K)$ is price of European call with strike price K and $f = f_X$ is risk neutral probability density function for X at time T, then $C(K) = e^{-rT} \int_{-\infty}^{\infty} f(x) \max\{0, x - K\} dx$.

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- Differentiating under the integral, we find that

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- We can look up $C(K)$ for a given stock symbol (say GOOG) and expiration time T at cboe.com and work out approximately what F_X and hence f_X must be.

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- "Implied volatility" is the value of σ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.
- If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices. In practice, when the implied volatility is viewed as a function of strike price (sometimes called the "volatility smile"), it is not constant.

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- **Replicating portfolio point of view:** in the simple binary tree models (or continuum Brownian models), we can transfer money back and forth between the stock and the risk free asset to ensure our wealth at time T equals the option payout. Option price is required initial investment, which is risk neutral expectation of payout. "True probabilities" are irrelevant.

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- **Where arguments for assumption break down:** Fluctuation sizes vary from day to day. Prices can have big jumps.

Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.
- **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large T.
- **Replicating portfolio point of view:** in the simple binary tree models (or continuum Brownian models), we can transfer money back and forth between the stock and the risk free asset to ensure our wealth at time T equals the option payout. Option price is required initial investment, which is risk neutral expectation of payout. "True probabilities" are irrelevant.
- **Where arguments for assumption break down:** Fluctuation sizes vary from day to day. Prices can have big jumps.
- **Fixes:** variable volatility, random interest rates, Lévy jumps....