18.600: Lecture 35

Martingales and the optional stopping theorem

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Outline

Martingales and stopping times

Optional stopping theorem

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- "The expected price tomorrow is the price today."
- If you are given a mathematical description of a process $X_0, X_1, X_2, \dots$ then how can you check whether it is a martingale?
- Consider all of the information that you know after having seen $X_0, X_1, \dots, X_n$. Then try to figure out what additional (not yet known) randomness is involved in determining X_{n+1}. Use this to figure out the conditional expectation of X_{n+1}, and check to see whether this is always equal to the known X_n value.

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- Say that T is a **stopping time** if the event that $T = n$ depends only on the values X_i for $i \leq n$. In other words, the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.

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- Answer: yes.
- What if each A_i is 1.01 with probability .5 and .99 with probability .5 and we write $X_0 = 1$ and $X_n = \prod_{i=1}^n A_i$ for $n > 0$? Then is X_n a martingale?

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- Answer: yes.
- These are two classic martingale examples: a sum of independent random variables (each with mean zero) and a product of independent random variables (each with mean one).

Stopping time examples

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- Which of the following is a stopping time?
  1. The smallest T for which $|X_T| = 50$
  2. The smallest T for which $X_T \in \{-10, 100\}$
  3. The smallest T for which $X_T = 0$.
  4. The T at which the X_n sequence achieves the value 17 for the 9th time.
  5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which X_T is largest.
  6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$.

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- Answer: first four, not last two.

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- Precisely, if you buy the asset at some time and adopt any strategy at all for deciding when to sell it, then the expected price at the time you sell is the price you originally paid.
- If market price is a martingale, you cannot make money in expectation by "timing the market."

Doob's Optional Stopping Theorem: statement

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- Theorem can be proved by induction if *stopping time* T is bounded. Unbounded T requires a limit argument. (This is where boundedness of martingale is used.)

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- But what about interest, risk premium, etc.?
- According to the **fundamental theorem of asset pricing**, the discounted price $\frac{X(n)}{A(n)}$, where A is a risk-free asset, is a martingale with respected to **risk neutral probability**. More on this next lecture.

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- More generally if Y_i are any random variables, the sequence $E[X], E[X|Y_1], E[X|Y_1, Y_2], E[X|Y_1, Y_2, Y_3], \ldots$ is a martingale.

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- Call me!!! I love you! Alice 0

More conditional probability martingale examples

- Example: let C be the amount of oil available for drilling under a particular piece of land. Suppose that ten geological tests are done that will ultimately determine the value of C. Let C_n be the **conditional expectation** of C *given* the outcome of the first n of these tests. Then the sequence $C_0, C_1, C_2, \ldots, C_{10} = C$ is a martingale.

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- Let A_i be my best guess at the probability that a basketball team will win the game, given the outcome of the first i minutes of the game. Then (assuming some "rationality" of my personal probabilities) A_i is a martingale.

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- **Question:** In low-volume market, might market manipulators (bidding up prices to make their candidates look better) overwhelm statistical arbitrageurs? If so, more money available for arbitrageurs who hang around.
- **Evidence for inefficiency:** price discrepancies, long shot bias.

Examples

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- What is the probability that it goes down to 45 then up to 55 then down to 45 then up to 55 again — all before reaching either 0 or 100?