# 18.600: Lecture 33

# Markov Chains

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# Outline

Markov chains

Examples

Ergodicity and stationarity

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Ergodicity and stationarity

# Markov chains

- Consider a sequence of random variables $X_0, X_1, X_2, \dots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \dots, M\}$.

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- Sequence is called a **Markov chain** if we have a fixed collection of numbers $P_{ij}$ (one for each pair $i, j \in \{0, 1, \ldots, M\}$) such that whenever the system is in state $i$, there is probability $P_{ij}$ that system will next be in state $j$.

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- Kind of an "almost memoryless" property. Probability distribution for next state depends only on the current state (and not on the rest of the state history).

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- Given that it is rainy today, how many days to I expect to have to wait to see a sunny day?
- Given that it is sunny today, how many days to I expect to have to wait to see a rainy day?
- Over the long haul, what fraction of days are sunny?

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- For this to make sense, we require $P_{ij} \geq 0$ for all $i, j$ and $\sum_{j=0}^{M} P_{ij} = 1$ for each $i$. That is, the rows sum to one.

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- If $A$ is the one-step transition matrix, then $A^n$ is the $n$-step transition matrix.

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- Answer: state evolution is deterministic.

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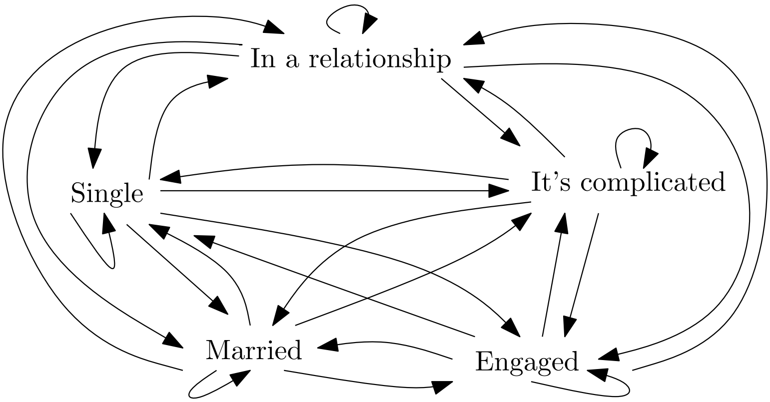
- Can compute $A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix}$

# Does relationship status have the Markov property?



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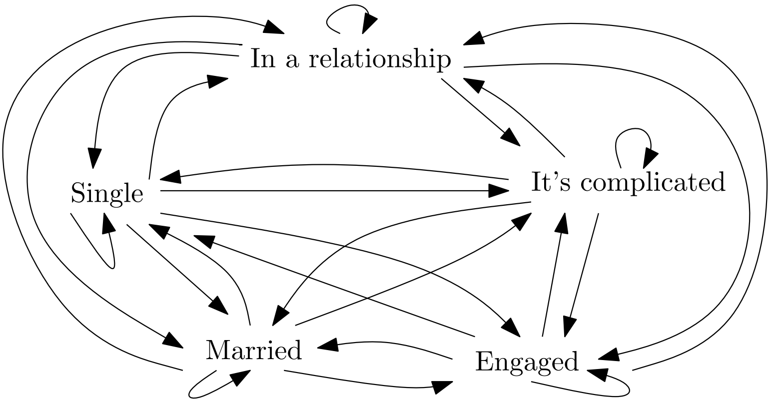


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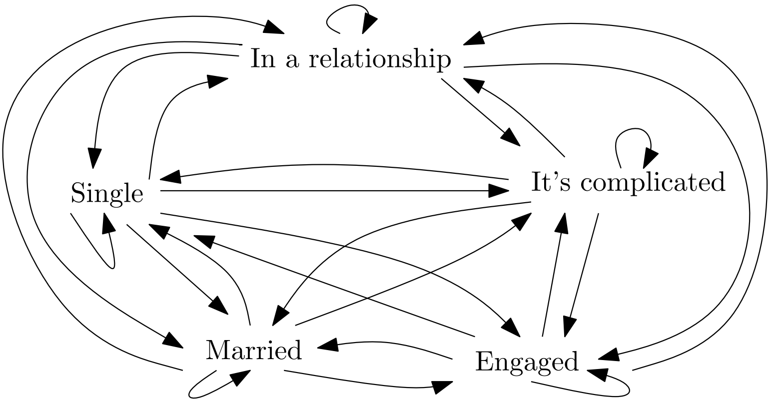


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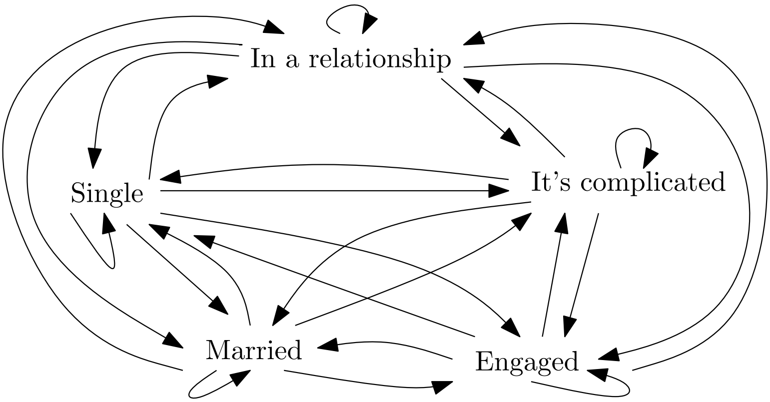


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- Not true... Can we make a better model with more states?

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- Turns out that if chain has this property, then $\pi_j := \lim_{n\to\infty} P_{ij}^{(n)}$ exists and the $\pi_j$ are the unique non-negative solutions of $\pi_j = \sum_{k=0}^M \pi_k P_{kj}$ that sum to one.

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- One can solve the system of linear equations $\pi_j = \sum_{k=0}^M \pi_k P_{kj}$ to compute the values $\pi_j$. Equivalent to considering $A$ fixed and solving $\pi A = \pi$. Or solving $(A - I)\pi = 0$. This determines $\pi$ up to a multiplicative constant, and fact that $\sum \pi_j = 1$ determines the constant.

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- If $A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$, then we know

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- This means that $.5\pi_0 + .2\pi_1 = \pi_0$ and $.5\pi_0 + .8\pi_1 = \pi_1$ and we also know that $\pi_0 + \pi_1 = 1$. Solving these equations gives $\pi_0 = 2/7$ and $\pi_1 = 5/7$, so $\pi = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix}$.

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- Recall that $A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix} \approx \begin{pmatrix} 2/7 & 5/7 \\ 2/7 & 5/7 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix}$