# 18.600: Lecture 27

# Moment generating functions and characteristic functions

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# Outline

Moment generating functions

Characteristic functions

Continuity theorems and perspective

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Characteristic functions

Continuity theorems and perspective

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- If $b > 0$ and $t > 0$ then $E[e^{tX}] \geq E[e^{t\min\{X,b\}}] \geq P\{X \geq b\}e^{tb}$.
- If $X$ takes both positive and negative values with positive probability then $M(t)$ grows at least exponentially fast in $|t|$ as $|t| \to \infty$.

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- Another way to think of this: write $e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots$.
- Taking expectations gives $E[e^{tX}] = 1 + tm_1 + \frac{t^2m_2}{2!} + \frac{t^3m_3}{3!} + \ldots$, where $m_k$ is the $k$th moment. The $k$th derivative at zero is $m_k$.

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- By independence, $M_Z(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$ for all $t$.

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- In other words, adding independent random variables corresponds to multiplying moment generating functions.

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- Answer: $M_X^n$. Follows by repeatedly applying formula above.
- This a big reason for studying moment generating functions. It helps us understand what happens when we sum up a lot of independent copies of the same random variable.

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- Latter answer is the special case of $M_Z(t) = M_X(t)M_Y(t)$ where $Y$ is the constant random variable $b$.

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- What if $X$ is Poisson with parameter $\lambda > 0$?
- Answer: $M_X(t) = E[e^{tx}] = \sum_{n=0}^{\infty} \frac{e^{tn}e^{-\lambda}\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{-\lambda}e^{\lambda e^t} = \exp[\lambda(e^t - 1)]$.

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- We know that if you add independent Poisson random variables with parameters $\lambda_1$ and $\lambda_2$ you get a Poisson random variable of parameter $\lambda_1 + \lambda_2$. How is this fact manifested in the moment generating function?

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- Answer: $Z$ has same law as $\sigma X + \mu$, so $M_Z(t) = M(\sigma t) e^{\mu t} = \exp\{\frac{\sigma^2 t^2}{2} + \mu t\}$.

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- Then $Z$ has the law of a sum of $n$ independent copies of $X$. So $M_Z(t) = M_X(t)^n = \left(\frac{\lambda}{\lambda-t}\right)^n$.

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- $M_X(t) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \lambda \int_0^\infty e^{-(\lambda-t)x}dx = \infty$ if $t \geq \lambda$.

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- Answer: $M_X(0) = 1$ (as is true for any $X$) but otherwise $M_X(t)$ is infinite for all $t \neq 0$.
- Informal statement: moment generating functions are not defined for distributions with fat tails.

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- And if $X$ has an $m$th moment then $E[X^m] = i^m\phi_X^{(m)}(0)$.

# Characteristic functions

- Let $X$ be a random variable.
- The **characteristic function** of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. Like $M(t)$ except with $i$ thrown in.
- Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.
- Characteristic functions are similar to moment generating functions in some ways.
- For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$.
- And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.
- And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- But characteristic functions have a distinct advantage: they are always well defined for all $t$ even if $f_X$ decays slowly.

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Continuity theorems and perspective

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- Moment generating functions are central to so-called *large deviation theory* and play a fundamental role in statistical physics, among other things.
- Characteristic functions are *Fourier transforms* of the corresponding distribution density functions and encode "periodicity" patterns. For example, if $X$ is integer valued, $\phi_X(t) = E[e^{itX}]$ will be 1 whenever $t$ is a multiple of $2\pi$.

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- **Moment generating analog:** if moment generating functions $M_{X_n}(t)$ are defined for all $t$ and $n$ and $\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$ for all $t$, then $X_n$ converge in law to $X$.