18.600: Lecture 20

Exponential random variables

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Outline

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Minimum of independent exponentials

Memoryless property

Relationship to Poisson random variables

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- Thus $P\{X < a\} = 1 - e^{-\lambda a}$ and $P\{X > a\} = e^{-\lambda a}$.
- Formula $P\{X > a\} = e^{-\lambda a}$ is very important in practice.

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- $E[X^0] = E[1] = 1$, $E[X] = 1/\lambda$, $E[X^2] = 2/\lambda^2$, $E[X^n] = n!/\lambda^n$.
- If $\lambda = 1$, the $E[X^n] = n!$. Could take this as definition of $n!$. It makes sense for $n = 0$ and for non-integer n.
- Variance: $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = 1/\lambda^2$.

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- Note: $X > a$ if and only if $X_1 > a$ *and* $X_2 > a$.
- X_1 and X_2 are independent, so $P\{X > a\} = P\{X_1 > a\}P\{X_2 > a\} = e^{-\lambda_1 a}e^{-\lambda_2 a} = e^{-\lambda a}$.

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- X_1 and X_2 are independent, so $P\{X > a\} = P\{X_1 > a\}P\{X_2 > a\} = e^{-\lambda_1 a}e^{-\lambda_2 a} = e^{-\lambda a}$.
- If $X_1, \ldots, X_n$ are independent exponential with $\lambda_1, \ldots \lambda_n$, then $\min\{X_1, \ldots X_n\}$ is exponential with $\lambda = \lambda_1 + \ldots + \lambda_n$.

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- Thus, conditional law of $X - b$ *given* that $X > b$ is same as the original law of X.

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- Given that the first 5 tosses are all tails, there is conditionally a .5 chance we get our first heads on the 6th toss, a .25 chance on the 7th toss, etc.
- Despite our having had five tails in a row, our expectation of the amount of time remaining until we see a heads is the same as it originally was.

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- **Bob:** No, that's your mistake. You should never assume that, because maybe somebody tampered with the coin.

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- How about an additional four weeks? Ten weeks?

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- Alice: you need assumptions to convert stories into math.
- Bob: good to question assumptions.

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- Claim: T_1 is exponential with parameter $n\lambda$.
- Claim: T_2 is exponential with parameter $(n-1)\lambda$.
- And so forth. $E[T] = \sum_{i=1}^n E[T_i] = \lambda^{-1} \sum_{j=1}^n \frac{1}{j}$ and (by independence) $\mathrm{Var}[T] = \sum_{i=1}^n \mathrm{Var}[T_i] = \lambda^{-2} \sum_{j=1}^n \frac{1}{j^2}$.

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- We actually did this already in the lecture on Poisson point processes. You can break the interval $[0, t]$ into n equal pieces (for very large n), let X_k be number of events in kth piece, use memoryless property to argue that the X_k are independent.
- When n is large enough, it becomes unlikely that any interval has more than one event. Roughly speaking: each interval has one event with probability $\lambda t/n$, zero otherwise.
- Take $n \to \infty$ limit. Number of events is Poisson λt.