# 18.600: Lecture 2

# Multinomial coefficients and more counting problems

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# Outline

Multinomial coefficients

Integer partitions

More problems

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- How many 8-letter sequences with 3 $A$'s, 2 $B$'s, and 3 $C$'s?
- Answer: $8!/(3!2!3!)$. Same as other problem. Imagine 8 "slots" for the letters. Choose 3 to be $A$'s, 2 to be $B$'s, and 3 to be $C$'s.

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- Answer $\binom{n}{n_1, n_2, \ldots, n_r} := \frac{n!}{n_1! n_2! \ldots n_r!}$.

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- Generally, $(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$, because there are $\binom{n}{k}$ sequences with $k$ $A$'s and $(n - k)$ $B$'s.

# How about trinomials?

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- What is the sum of the coefficients in this expansion? What is the combinatorial interpretation of coefficient of, say, $ABC^2$?
- Answer $81 = (1 + 1 + 1)^4$. $ABC^2$ has coefficient 12 because there are 12 length-4 words have one $A$, one $B$, two $C$'s.

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- Pascal's triangle gives coefficients in binomial expansions. Is there something like a "Pascal's pyramid" for trinomial expansions?

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- Because there is a consensus among MIT faculty that 0! should be 1.

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- Answer: $\binom{n+k-1}{n}$. Represent partition by $k-1$ bars and $n$ stars, e.g., as $**|**||****|*$.

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- $\binom{179}{90} = \binom{179}{89}$

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- How many hands have either 3 or 4 cards in each suit?
- Need three 3-card suits, one 4-card suit, to make 13 cards total. Answer is $4\binom{13}{3}^3\binom{13}{4}$