18.600: Lecture 11

Binomial random variables and repeated trials

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Outline

Bernoulli random variables

Properties: expectation and variance

More problems

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- $1 = 1^n = (p+q)^n = \sum_{k=0}^n \binom{n}{k}p^k q^{n-k}$.
- Number of heads is **binomial random variable with parameters** (n, p).

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- Probability mass function for X can be computed using the 6th row of Pascal's triangle.
- If coin is biased (comes up heads with probability $p \neq 1/2$), we can still use the 6th row of Pascal's triangle, but the probability that $X = i$ gets multiplied by $p^i(1-p)^{n-i}$.

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- Let $n = 100$. Compute the probability that nobody was born on a Tuesday.
- What is the probability that exactly 15 people were born on a Tuesday?

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- For example, replace the 5th row $(1, 5, 10, 10, 5, 1)$ by $(0, 5, 20, 30, 20, 5)$. Does this remind us of an earlier row in the triangle?
- Perhaps the prior row $(1, 4, 6, 4, 1)$?

Useful Pascal's triangle identity

- Recall that $\binom{n}{i} = \frac{n\times(n-1)\times\ldots\times(n-i+1)}{i\times(i-1)\times\ldots\times(1)}$. This implies a simple but important identity: $i\binom{n}{i} = n\binom{n-1}{i-1}$.

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- Using this identity (and $q = 1-p$), we can write

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- Rewrite this as $E[X] = np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{(i-1)} q^{(n-1)-(i-1)}$.
- Substitute $j = i - 1$ to get

$$E[X] = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j} = np(p+q)^{n-1} = np.$$

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- Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^n E[X_j] = \sum_{j=1}^n p = np.$$

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- Thus $E[X^k] = npE[(Y+1)^{k-1}]$ where Y is binomial with parameters $(n-1, p)$.

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- This is n times the variance you'd get with a single coin. Coincidence?

Compute variance with decomposition trick

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- You invite 50 friends to a party. Each one, independently, has a $1/3$ chance of showing up. What is the probability that more than 25 people will show up?