# 18.600: Lecture 11 Binomial random variables and repeated trials

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Bernoulli random variables

### Properties: expectation and variance

More problems

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- Probability mass function for X can be computed using the 6th row of Pascal's triangle.
- If coin is biased (comes up heads with probability p ≠ 1/2), we can still use the 6th row of Pascal's triangle, but the probability that X = i gets multiplied by p<sup>i</sup>(1 − p)<sup>n−i</sup>.

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- Let n = 100. Compute the probability that nobody was born on a Tuesday.
- What is the probability that exactly 15 people were born on a Tuesday?

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- Perhaps the prior row (1, 4, 6, 4, 1)?

► Recall that (<sup>n</sup><sub>i</sub>) = <u>n×(n-1)×...×(n-i+1)</u>. This implies a simple but important identity: i(<sup>n</sup><sub>i</sub>) = n(<sup>n-1</sup><sub>i-1</sub>).

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- Using this identity (and q = 1 p), we can write

$$E[X] = \sum_{i=0}^{n} i\binom{n}{i} p^{i} q^{n-i} = \sum_{i=1}^{n} n\binom{n-1}{i-1} p^{i} q^{n-i}$$

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- ▶ Substitute j = i − 1 to get

$$E[X] = np \sum_{j=0}^{n-1} {n-1 \choose j} p^j q^{(n-1)-j} = np(p+q)^{n-1} = np.$$

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- Note that  $E[X_j] = p \cdot 1 + (1-p) \cdot 0 = p$  for each j.
- Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} p = np.$$

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► Thus  $E[X^k] = npE[(Y+1)^{k-1}]$  where Y is binomial with parameters (n-1, p).

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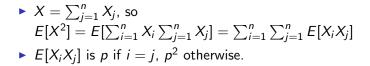
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- This is n times the variance you'd get with a single coin. Coincidence?

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, so  
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- $\sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]$  has *n* terms equal to *p* and (n-1)n terms equal to  $p^2$ .

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$$Var[X] = E[X^2] - E[X]^2 = np - np^2 = np(1-p) = npq.$$

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- You invite 50 friends to a party. Each one, independently, has a 1/3 chance of showing up. What is the probability that more than 25 people will show up?