18.440 PROBLEM SET SIX, DUE NOVEMBER 6

A. FROM TEXTBOOK CHAPTER FIVE:

1. Problems: 41.
2. Theoretical Exercises: 18, 25, 26

B. FROM TEXTBOOK CHAPTER SIX:

1. Problems: 12, 30, 33.
2. Theoretical Exercises: 10, 24
3. Self-Test Problems and Exercises: 20.

C.

1. Suppose that $X_1, X_2, \ldots X_n$ are independent random normal variables with $\mathbb{E}X_i = \mu_i$ and $\mathrm{Var}X_i = \sigma_i^2$. Write down the probability density function for the random variable $Y = \sum_{i=1}^n X_i$.
2. Suppose that X_1, X_2, and X_3 are independent uniform random variables on the interval $[0, 1]$. Write down (and sketch) the probability density function for $X_1 + X_2 + X_3$.
3. Suppose that X is a random variable with a continuous probability density function f. Can you always find a function g such that $g(X)$ is a uniform random variable on $[0, 1]$? If so, give an explicit formula for g (in terms of f) — if not, give a counterexample.

D. (*)

1. Suppose that X_i are independent standard normal random variables. Show that $\sum_{i=1}^{n} X_i$ has the same probability density function as $\sqrt{n}X_1$.

2. Suppose that X_i are independent Cauchy random variables. Show that $Y = \sum_{i=1}^{n} X_i$ has the same probability density function as nX_1.

3. For what real values of α does there exist a probability density function f that is symmetric about zero (i.e., $f(x) = f(-x)$) such that if X_i are independently distributed with density function f, then $Y = \sum_{i=1}^{n} X_i$ has the same probability density function as $n^{1/\alpha}X_i$, for all n? (By the above, at least $\alpha = 1$ and $\alpha = 2$ are possible.)

4. Give an explicit description of f. (Hint one: what can you say about the Fourier transform of f? Hint two: after making an honest effort using hint one, try googling "stable distribution.")

5. Look up "stable Levy process" and "Brownian motion. " Explain what these objects are and how they are related to your answers above.