18.440 PROBLEM SET FOUR, DUE OCTOBER 9

A. FROM TEXTBOOK CHAPTER FOUR:

1. Problems: 11, 21, 30, 46, 50.
2. Theoretical Exercises: 4, 5, 13, 17.
3. Self-Test Problems and Exercises: 3, 7.

B. Define the covariance $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

1. Check that $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$, that $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$, and that $\mathrm{Cov}(\cdot,\cdot)$ is a bilinear function of its arguments. That is, if one fixes one argument then it is a linear function of the other. For example, if we fix the second argument then for real constants a and b we have $\mathrm{Cov}(aX+bY,Z) = a\mathrm{Cov}(X,Z) + b\mathrm{Cov}(Y,Z)$.
2. If $\mathrm{Cov}(X_i,X_j) = ij$, find $\mathrm{Cov}(X_1 - X_2, X_3 - 2X_4)$.
3. If $\mathrm{Cov}(X_i,X_j) = ij$, find $\mathrm{Var}(X_1 + 2X_2 + 3X_3)$.

C. Let's suppose that instead of maximizing her expected wealth $\mathbb{E}[W]$, Jill always maximizes $\mathbb{E}[U(W)]$ where $U(x) = -(x-x_0)^2$ where x_0 is a large positive number. In the language of economics, Jill has a *quadratic utility function*. (It may seem a little strange that Jill starts to become less happy when her wealth exceeds x_0, but let us assume x_0 is large enough so that this event is unlikely.) Jill currently has W_0 dollars. You propose to sample a random variable X (with mean μ and variance σ^2) and to give her X dollars (she will lose money if X is negative) so that her new wealth becomes $W = W_0 + X$.

1. Show that $\mathbb{E}[U(W)]$ depends on μ and σ^2 (but not on any other information about the probability distribution of X) and compute $\mathbb{E}[U(W)]$ as a function of x_0, W_0, μ, σ^2.
2. Show that given μ, Jill would prefer for σ^2 to be as small as possible. (One sometimes refers to σ as *risk* and says that Jill is *risk averse*.)
3. Suppose that $X = \sum_{i=1}^n a_i X_i$ where a_i are fixed constants and the X_i are random variables with $\mathbb{E}[X_i] = \mu_i$ and $\mathrm{Cov}[X_i,X_j] = \sigma_{ij}$. Show that in this case $\mathbb{E}[U(W)]$ depends on the μ_i and the σ_{ij} (but not on any other information about the joint probability distributions of the X_i) and compute $\mathbb{E}[U(W)]$.

4. Suppose that D is the random variable corresponding to a fair die roll. Suppose that investment one returns $X_1 = D - 3$ dollars, investment two returns $X_2 = D^2 - 4$ dollars, and investment three returns $X_3 = 1 + 2(-1)^D$ dollars. Compute the corresponding three-by-three matrix σ_{ij} and vector μ_j.

5. Read the Wikipedia article on "Modern Portfolio Theory". (The article mentions assumptions that the X_i are normally distributed and that future variances can be estimated from historical volatility. These assumptions are *not* necessary for the problem of the computing mean and variance of X when the joint probability distributions of the X_i are actually *known*.) Summarize what you learned in two or three sentences.

6. (*) Consider the set of all random variables of the form $\sum_{i=1}^{3} a_i X_i$ where $\sum_{i=1}^{3} a_i = 1$. Sketch the set of all possible (σ, μ) pairs arising from elements of this set. This is sometimes called the *Markowitz bullet*. Sketch the set obtained if one adds the assumption that the a_i are between 0 and 1.

D. (*) *Create your own paradox!* Let X be any integer-valued random variable for which $\sum_{k=0}^{\infty} kP\{X = k\} = \infty$ and $\sum_{k=-\infty}^{0} kP\{X = k\} = -\infty$.

1. Show that there is a function $f : \mathbb{Z} \to \mathbb{Z}$ with the following properties:
   (a) For each integer y, there are only finitely many integers x with $f(x) = y$.
   (b) If Y is the random variable $f(X)$, then for each y such that $P\{Y = y\} > 0$, the following holds: given that $Y = y$, the conditional expectation of X is greater than 10^9.
   (c) Show that there is a similar function g and random variable $Z = g(X)$ such that for each z such that $P\{Z = z\} > 0$, the following holds: given that $Z = z$, the conditional expectation of X is less than -10^9.

2. Declare "I propose that Jill gives X dollars to Beth". (Note that Jill collects money from Beth if X is negative.) Then sample X and show Beth Y and show Jill Z, while keeping X itself a secret. Argue that if Jill and Beth are expectation maximizers (as in the previous problem set), they would both gratefully accept your proposal. What if instead of showing Beth Y and Jill Z you showed Jill Y and Beth Z?

HINTS:

1. First imagine that you have a sequence of coins indexed by $\mathbb{Z}$, with the kth coin being worth $kP\{X = k\}$ dollars. (Some coins have negative worth.) Show that you can divide all the coins into sacks (finitely many coins per sack) in such a way that each sack is worth at least 10^9 dollars. Show that you can also divide the coins into sacks in such a way that each sack is worth at most -10^9 dollars.

2. Before dealing with the first hint, tackle an easier problem: imagine that every negative integer corresponds to a green apple and every positive integer to a red apple. Show that you can divide the apples into sacks in such a way that every sack has 10^9 red apples and one green apple. Show that you can also divide them in such a way that every sack has 10^9 green apples and one red apple.

3. Argue that the paradoxes above and the paradoxes of the previous problem set are really nothing more than cleverly described versions of the "apple paradox." And this in turn can be derived from more pedestrian "paradoxes" (such as the fact that the set of all integers is in bijective correspondences with the set of integer multiples of 10^9).