18.440 PROBLEM SET THREE, DUE OCTOBER 2

A. FROM TEXTBOOK CHAPTER THREE:

1. Problems: 31, 33, 43, 74.
2. Theoretical Exercises: 5, 10.
3. Self-Test Problems and Exercises: 8, 14, 22.

B. Two unfair dice are tossed. Let $p_{i,j}$, for i and j in $\{1, 2, 3, 4, 5, 6\}$, denote the probability that the first die comes up i and the second j. Suppose that for any i and j in $\{1, 2, 3, 4, 5, 6\}$ the event that the first die comes up i is independent of the event that the second die comes up j. Show that this independence implies that, as a 6 by 6 matrix, $p_{i,j}$ has rank one (i.e., show that there is some column of the matrix such that each of the other five column vectors is a constant multiple of that one).

C. The following is one formulation of a famous "two envelope" paradox. Jill is a money-loving individual who, given two options, invariably chooses the one that gives her the most money in expectation. One day Harry, a trusted (and capable of delivering) individual, offers her the following deal as a gift. He will secretely toss a fair coin until the first time that it comes up tails. If there are n heads before the first tails, he will place 10^n dollars in one envelope and 10^{n+1} dollars in the second envelope. (Thus, the probability that one envelope has 10^n dollars and the other has 10^{n+1} dollars is 2^{-n-1} for $n \geq 0$.) Harry will then hand Jill the pair of envelopes (randomly ordered, indistinguishable from the outside) and invite her to choose one. After Jill chooses an envelope she will be allowed to open it. Once she does, she will be allowed to either keep the money in the first envelope or switch to the second envelope and keep whatever amount of money is in the second envelope. However, if she decides to switch envelopes, she has to pay a one dollar "switching fee."

1. If Jill finds 100 dollars in the first envelope she opens, what is the conditional probability that the other envelope contains 1000 dollars? What is the conditional probability that the other envelope contains 10 dollars?
2. If Jill finds 100 dollars in the first envelope she opens, how much money does Jill expect to win from the game if she does not switch envelopes? (Answer: 100 dollars.) How much does she expect to win (net, after the switching fee) if she *does* switch envelopes?

3. Generalize the answers above to the case that the first envelope contains 10^n dollars (for $n \geq 0$) instead of 100.

4. (*) Jill concludes from the above that, no matter what she finds in the first envelope, she will expect to earn more money if she switches envelopes and pays the one dollar switching fee. This strikes Jill as a bit odd. If she knows she will always switch envelopes, why doesn't she just take the second envelope first and avoid the envelope switching fee? How can she be maximizing her expected wealth if she spends an unnecessary "switching fee" dollar no matter what? How does one resolve this apparent paradox?

5. (*) Enterprising Ernie views every mathematical puzzle as a money-making opportunity. (For example, he has been known to charge people 51 cents to play a card game that pays 50 cents in expectation regardless of strategy.) Ernie knows that both Jill and Beth are capable of paying arbitrarily large sums, but that both Jill and Beth, given a choice between two options, will always maximize their own expected payoffs. He invites Jill and her like-minded friend Beth to play the following game with him. Ernie chooses numbers 10^n and 10^{n+1} from the probability distribution above, writes them on two slips of paper, and puts one slip in each of two envelopes. He then gives one envelope to Jill and one to Beth. Let B be the amount in Beth's envelope and J the amount in Jill's envelope. Jill and Beth observe only the numbers in their own envelopes. Ernie tells Beth that if she gives him $B + 1$ dollars, he will give her J dollars. He tells Jill that if she gives him $J + 1$ dollars he will give her B dollars. Ernie is confident that both Jill and Beth will reason that (conditioned on what they have seen on their own envelopes) the transaction will make them money in expectation. Thus, Ernie will receive $B + 1 + J + 1$ dollars while paying out only $B + J$ — a sure two dollar profit for Ernie. Is this game a good deal for all three players? Explain.