18.440 PRACTICE PROBLEM SET: TO BE DONE (BUT NOT HANDED IN) BY NOVEMBER 13

A. FROM TEXTBOOK CHAPTER SIX:

1. Self-Test Problems and Exercises: 5, 7, 12, 14, 15, 16, 19.

B. FROM TEXTBOOK CHAPTER SEVEN:

1. Theoretical Exercises: 7, 8, 24, 27, 39, 46, 47, 49, 50, 53, 55.
2. Self-Test Problems and Exercises: 3, 5, 14, 26, 27, 29, 30.

C. Suppose that $X_1, X_2, \ldots, X_n$ are bounded random variables for which

$$\mathbb{E}[X_k | X_1, X_2, \ldots, X_{k-1}] = X_{k-1}$$

for $2 \leq k \leq n$. (This kind of sequence of random variables is called a *martingale*.) Show the following:

1. $\mathbb{E}X_n = \mathbb{E}X_1$.
2. If $1 \leq j < k \leq n$ then $\mathbb{E}[X_k | X_j] = X_j$.
3. The increments $I_j := X_j - X_{j-1}$, defined for $2 \leq j \leq n$, have mean zero and are pairwise uncorrelated: thus, $\mathbb{E}I_j I_k = 0$ for $j \neq k$.
4. $\mathrm{Var}(X_n) = \mathrm{Var}(X_1) + \sum_{j=2}^n \mathrm{Var}(I_j)$.
5. $\mathrm{Var}(X_n) = \mathrm{Var}(X_1) + \sum_{j=2}^n \mathbb{E}[\mathrm{Var}(X_j | X_{j-1})]$.
6. Consider the special case in which $X_k = \prod_{j=1}^k Y_j$ where the Y_j are independent random variables that each assume the value .99 with probability 1/2 and 1.01 with probability 1/2. Show that $\mathbb{E}X_{1000} = 1$ and estimate the probability that $X_{1000} > 1$. (Hint: take logs and use the central limit theorem for coin tosses.) Note: this kind of sequence is often used in finance to model asset prices that go up or down by a random percentage during each time incremeent.