18.440: Lecture 4

Axioms of probability and inclusion-exclusion

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Outline

Axioms of probability

Consequences of axioms

Inclusion exclusion
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Axioms of probability

Consequences of axioms

Inclusion exclusion
Axioms of probability

- $P(A) \in [0, 1]$ for all $A \subset S$. 

18.440 Lecture 4
Axioms of probability

- \( P(A) \in [0, 1] \) for all \( A \subset S \).
- \( P(S) = 1 \).
Axioms of probability

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- Finite additivity: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.
Axioms of probability

- $P(A) \in [0, 1]$ for all $A \subset S$.
- $P(S) = 1$.
- Finite additivity: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.
- Countable additivity: $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \emptyset$ for each pair $i$ and $j$. 
Neurological: When I think “it will rain tomorrow” the “truth-sensing” part of my brain exhibits 30 percent of its maximum electrical activity.
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Frequentist: $P(A)$ is the fraction of times $A$ occurred during the previous (large number of) times we ran the experiment.

Market preference ("risk neutral probability"): $P(A)$ is price of contract paying dollar if $A$ occurs divided by price of contract paying dollar regardless.
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What if personal belief function doesn’t satisfy axioms?
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Consider an $A$-contract (pays 10 if candidate $A$ wins election) a $B$-contract (pays 10 dollars if candidate $B$ wins) and an $A$-or-$B$ contract (pays 10 if either $A$ or $B$ wins).
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Professional response: “I fully understand and respect your opinions. In fact, let’s do some business. You sell me an \( A \) contract and a \( B \) contract for 1.50 each, and I sell you an \( A \)-or-\( B \) contract for 6.50.”
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Axioms breakdowns are money-making opportunities.
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Frequentist: $P(A)$ is the fraction of times $A$ occurred during the previous (large number of) times we ran the experiment. Seems to satisfy axioms... 

Market preference (“risk neutral probability”): $P(A)$ is price of contract paying dollar if $A$ occurs divided by price of contract paying dollar regardless. Seems to satisfy axioms, assuming no arbitrage, no bid-ask spread, complete market... 

Personal belief: $P(A)$ is amount such that I’d be indifferent between contract paying 1 if $A$ occurs and contract paying $P(A)$ no matter what. Seems to satisfy axioms with some notion of utility units, strong assumption of “rationality”...
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Consequences of axioms

Inclusion exclusion
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Inclusion exclusion
We will sometimes write $AB$ to denote the event $A \cap B$. 
Consequences of axioms

- Can we show from the axioms that $P(A^c) = 1 - P(A)$?
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- Can we show from the axioms that $P(AB) \leq P(A)$?
- Can we show from the axioms that if $S$ contains finitely many elements $x_1, \ldots, x_k$, then the values $(P(\{x_1\}), P(\{x_2\}), \ldots, P(\{x_k\}))$ determine the value of $P(A)$ for any $A \subset S$?
Consequences of axioms

- Can we show from the axioms that \( P(A^c) = 1 - P(A) \)?
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- What \( k \)-tuples of values are consistent with the axioms?
People are told “Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.”
Famous 1982 Tversky-Kahneman study (see wikipedia)

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- They are asked: Which is more probable?
  - Linda is a bank teller.
  - Linda is a bank teller and is active in the feminist movement.
People are told “Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.”

They are asked: Which is more probable?
- Linda is a bank teller.
- Linda is a bank teller and is active in the feminist movement.

85 percent chose the second option.
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Could be correct using neurological/emotional definition. Or a “which story would you believe” interpretation (if witnesses offering more details are considered more credible).
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But axioms of probability imply that second option cannot be more likely than first.
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Inclusion exclusion
Imagine we have $n$ events, $E_1, E_2, \ldots, E_n$. How do we go about computing something like $P(E_1 \cup E_2 \cup \ldots \cup E_n)$? It may be quite difficult, depending on the application. There are some situations in which computing $P(E_1 \cup E_2 \cup \ldots \cup E_n)$ is a priori difficult, but it is relatively easy to compute probabilities of intersections of any collection of $E_i$. That is, we can easily compute quantities like $P(E_1 E_3 E_7)$ or $P(E_2 E_3 E_6 E_7 E_8)$. In these situations, the inclusion-exclusion rule helps us compute unions. It gives us a way to express $P(E_1 \cup E_2 \cup \ldots \cup E_n)$ in terms of these intersection probabilities.
Imagine we have \( n \) events, \( E_1, E_2, \ldots, E_n \).

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In these situations, the inclusion-exclusion rule helps us compute unions. It gives us a way to express \( P(E_1 \cup E_2 \cup \ldots \cup E_n) \) in terms of these intersection probabilities.
Inclusion-exclusion identity

Can we show from the axioms that
\[ P(A \cup B) = P(A) + P(B) - P(AB) \]?
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- How about
  \[ P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG) \]?
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- More generally,
  \[
P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \ldots \\
+ (-1)^{r+1} \sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1} E_{i_2} \ldots E_{i_r}) \\
+ \ldots + (-1)^{n+1} P(E_1 E_2 \ldots E_n).
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\]
- The notation \( \sum_{i_1 < i_2 < \ldots < i_r} \) means a sum over all of the \( \binom{n}{r} \) subsets of size \( r \) of the set \( \{1, 2, \ldots, n\} \).
Consider a region of the Venn diagram contained in exactly \( m > 0 \) subsets. For example, if \( m = 3 \) and \( n = 8 \) we could consider the region \( E_1 E_2 E_3^c E_4^c E_5 E_6^c E_7^c E_8^c \).
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This region is contained in three single intersections ($E_1$, $E_2$, and $E_5$). It’s contained in 3 double-intersections ($E_1 E_2$, $E_1 E_5$, and $E_2 E_5$). It’s contained in only 1 triple-intersection ($E_1 E_2 E_5$).
Inclusion-exclusion proof idea

Consider a region of the Venn diagram contained in exactly \( m > 0 \) subsets. For example, if \( m = 3 \) and \( n = 8 \) we could consider the region \( E_1 E_2 E_3^c E_4^c E_5 E_6^c E_7^c E_8^c \).

This region is contained in three single intersections (\( E_1, E_2, \) and \( E_5 \)). It’s contained in 3 double-intersections (\( E_1 E_2, E_1 E_5, \) and \( E_2 E_5 \)). It’s contained in only 1 triple-intersection (\( E_1 E_2 E_5 \)).

It is counted \( \binom{m}{1} - \binom{m}{2} + \binom{m}{3} + \ldots \pm \binom{m}{m} \) times in the inclusion exclusion sum.
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How many is that?
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How many is that?

Answer: 1. (Follows from binomial expansion of \( (1 - 1)^m \).)
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This region is contained in three single intersections (\( E_1, E_2, \) and \( E_5 \)). It’s contained in 3 double-intersections (\( E_1 E_2, E_1 E_5, \) and \( E_2 E_5 \)). It’s contained in only 1 triple-intersection (\( E_1 E_2 E_5 \)).

It is counted \( \binom{m}{1} - \binom{m}{2} + \binom{m}{3} + \ldots \pm \binom{m}{m} \) times in the inclusion exclusion sum.

How many is that?

Answer: 1. (Follows from binomial expansion of \((1 - 1)^m\).)

Thus each region in \( E_1 \cup \ldots \cup E_n \) is counted exactly once in the inclusion exclusion sum, which implies the identity.
Famous hat problem

- $n$ people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.
Famous hat problem

- $n$ people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.

- Inclusion-exclusion. Let $E_i$ be the event that $i$th person gets own hat.

\[ P(\bigcup_{i=1}^{n} E_i) = 1 - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \left( \begin{array}{c} n \end{array} \right) \left( \begin{array}{c} n \end{array} \right)^{n-k} \]

\[ \approx 1 - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \approx \frac{1}{e} \approx 0.36788 \]
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- Inclusion-exclusion. Let $E_i$ be the event that $i$th person gets own hat.
- What is $P(E_{i_1} E_{i_2} \ldots E_{i_r})$?
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- Answer: $\frac{(n-r)!}{n!}$.
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- Answer: $\frac{(n-r)!}{n!}$.
- There are $\binom{n}{r}$ terms like that in the inclusion exclusion sum.
- What is $\binom{n}{r} \frac{(n-r)!}{n!}$?
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  - There are $\binom{n}{r}$ terms like that in the inclusion exclusion sum.
  - What is $\binom{n}{r} \frac{(n-r)!}{n!}$?
  - Answer: $\frac{1}{r!}$. 

Inclusion-exclusion sum:

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots - \frac{1}{n!} \approx 1/e \approx .36788$$
Famous hat problem

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- Inclusion-exclusion. Let \( E_i \) be the event that \( i \)th person gets own hat.
- What is \( P(E_1 E_2 \ldots E_r) \)?
- Answer: \( \frac{(n-r)!}{n!} \).
- There are \( \binom{n}{r} \) terms like that in the inclusion exclusion sum.
- What is \( \binom{n}{r} \frac{(n-r)!}{n!} \)?
- Answer: \( \frac{1}{r!} \).
- \( P(\bigcup_{i=1}^{n} E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!} \)
Famous hat problem

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- Answer: $\frac{(n-r)!}{n!}$.
- There are $\binom{n}{r}$ terms like that in the inclusion-exclusion sum. What is $\binom{n}{r} \frac{(n-r)!}{n!}$?
- Answer: $\frac{1}{r!}$.
- $P(\bigcup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!}$
- $1 - P(\bigcup_{i=1}^n E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots \pm \frac{1}{n!} \approx 1/e \approx .36788$