# 18.440: Lecture 39

# Review: practice problems

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- When Alice showers, she first checks to see if at least one towel is present. If a towel is present, she dries off with that towel and returns it to the bathroom towel rack. Otherwise, she cheerfully retrieves both towels from the walk-in closet, then showers, dries off and leaves both towels on the rack.

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- **Problem:** describe towel-distribution evolution as a Markov chain and determine (over the long term) on what fraction of days Bob emerges from the shower to find no towel.

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- Bob finds no towel only if morning starts in state zero and Bob goes first. Over long term Bob finds no towel $\frac{2}{9} \times \frac{1}{2} = \frac{1}{9}$ fraction of the time.

# Optional stopping, martingales, central limit theorem

Suppose that $X_1, X_2, X_3, \dots$ is an infinite sequence of independent random variables which are each equal to 1 with probability $1/2$ and $-1$ with probability $1/2$. Let $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

- What is the the probability that $Y_n$ reaches $-25$ before the first time that it reaches 5?

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Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 1 with probability $1/2$ and $-1$ with probability $1/2$. Let $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

- What is the the probability that $Y_n$ reaches $-25$ before the first time that it reaches 5?
- Use the central limit theorem to approximate the probability that $Y_{9000000}$ is greater than 6000.

# Optional stopping, martingales, central limit theorem — answers

- $p_{-25}25 + p_5 5 = 0$ and $p_{-25} + p_5 = 1$. Solving, we obtain $p_{-25} = 1/6$ and $p_5 = 5/6$.

# Optional stopping, martingales, central limit theorem — answers

- $p_{-25}25 + p_5 5 = 0$ and $p_{-25} + p_5 = 1$. Solving, we obtain $p_{-25} = 1/6$ and $p_5 = 5/6$.
- One standard deviation is $\sqrt{9000000} = 3000$. We want probability to be 2 standard deviations above mean. Should be about $\int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

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  - $Y_n = \prod_{i=1}^n (X_i - 1)$

# Martingales

- Yes, no, yes, no.

# Calculations like those needed for Black-Scholes derivation

- Let $X$ be a normal random variable with mean 0 and variance 1. Compute the following (you may use the function $\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answers):

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  - $E[e^{3X-3}]$.

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  - $E[e^{3X-3}]$.
  - $E[e^X 1_{X \in (a,b)}]$ for fixed constants $a < b$.

# Calculations like those needed for Black-Scholes derivation – answers

$$
\begin{aligned}
E[e^{3X-3}] &= \int_{-\infty}^{\infty} e^{3x-3}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2-6x+6}{2}}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2-6x+9}{2}}e^{3/2}dx \\
&= e^{3/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-3)^2}{2}}dx \\
&= e^{3/2}
\end{aligned}
$$

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\begin{aligned}
E[e^X 1_{X\in(a,b)}] &= \int_a^b e^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \int_a^b e^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2x+1-1}{2}} dx \\
&= e^{1/2} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx \\
&= e^{1/2} \int_{a-1}^{b-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= e^{1/2}(\Phi(b-1) - \Phi(a-1))
\end{aligned}
$$

# Thanks for taking the course!

- **FRIENDLY FAREWELL:**
  Good night, good night! **Parting is such sweet sorrow**
  That I shall say good night till it be morrow.
  *Romeo And Juliet Act 2, Scene 2*

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- Morrow = May 20.

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- Morrow = May 20.
- **UNFRIENDLY FAREWELL:**
  Go make some new disaster,
  That's what I'm counting on.
  You're someone else's problem.
  Now I only want you gone.
  *Portal 2 Closing Song*

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- **PRACTICAL FAREWELL:**
  Consider 18.443 (statistics), 18.424 (entropy/information) 18.445 (Markov chains), 18.472 (math finance), 18.175 (grad probability), 18.176 (martingales, stochastic processes), 18.177 (special topics), 18.338 (random matrices), 18.466 (grad statistics), many non-18 courses. See you May 20.