Review: practice problems

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Eight athletic teams are ranked 1 through 8 after season one, and ranked 1 through 8 again after season two. Assume that each set of rankings is chosen uniformly from the set of 8! possible rankings and that the two rankings are independent. Let $N$ be the number of teams whose rank does not change from season one to season two. Let $N_+$ the number of teams whose rank improves by exactly two spots. Let $N_-$ be the number whose rank declines by exactly two spots. Compute the following:
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- $E[N]$, $E[N_+]$, and $E[N_-]$
- $\text{Var}[N]$
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- $E[N]$, $E[N_+]$, and $E[N_-]$
- $\text{Var}[N]$
- $\text{Var}[N_+]$
Let \( N_i \) be 1 if team ranked \( i \)th first season remains \( i \)th second seasons. Then 
\[
E[N] = E\left[\sum_{i=1}^{8} N_i\right] = 8 \cdot \frac{1}{8} = 1.
\]
Similarly, 
\[
E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4.
\]
Let $N_i$ be 1 if team ranked $i$th first season remains $i$th second seasons. Then $E[N] = E[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1$. Similarly, $E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4$.

$\text{Var}[N] = E[N^2] - E[N]^2$ and

$E[N^2] = E[\sum_{i=1}^{8} \sum_{j=1}^{8} N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2$. 
Let $N_i$ be 1 if team ranked $i$th first season remains $i$th second seasons. Then $E[N] = E[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1$. Similarly, $E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = \frac{3}{4}$.

$\text{Var}[N] = E[N^2] - E[N]^2$ and $E[N^2] = E[\sum_{i=1}^{8} \sum_{j=1}^{8} N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2$.

Let $N_i^+ \equiv 1$ if team ranked $i$th has rank improve to $(i - 2)$th for second seasons. Then $E[(N_+)^2] = E[\sum_{3=1}^{8} \sum_{3=1}^{8} N_+^i N_+^j] = 6 \cdot \frac{1}{8} + 30 \cdot \frac{1}{56} = \frac{9}{7}$, so $\text{Var}[N_+] = \frac{9}{7} - \left(\frac{3}{4}\right)^2$. 

18.440 Lecture 37
Roll ten dice. Find the conditional probability that there are exactly 4 ones, given that there are exactly 4 sixes.
Straightforward approach: \( P(A|B) = \frac{P(AB)}{P(B)} \).
Conditional distributions — answers

- Straightforward approach: \( P(A|B) = \frac{P(AB)}{P(B)} \).
- Numerator: is \( \binom{10}{4}\binom{6}{4}^2 \). Denominator is \( \frac{\binom{10}{4}^5}{6^{10}} \).
Straightforward approach: $P(A|B) = P(AB)/P(B)$.

Numerator: is $\frac{(\binom{10}{4})(\binom{6}{4})^2}{6^{10}}$. Denominator is $\frac{(\binom{10}{4})^5}{6^{10}}$.

Ratio is $\frac{(\binom{6}{4})4^2}{5^6} = \frac{(\binom{6}{4})(\frac{1}{5})^4(\frac{4}{5})^2}$. 
Conditional distributions — answers

- Straightforward approach: \( P(A|B) = \frac{P(AB)}{P(B)} \).
- Numerator: is \( \binom{10}{4}\binom{6}{4}4^2 \). Denominator is \( \binom{10}{4}\binom{5}{6} \).
- Ratio is \( \frac{\binom{6}{4}4^2}{5^6} = \frac{\binom{6}{4}}{\binom{6}{4}}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^2 \).
- Alternate solution: first condition on location of the 6’s and then use binomial theorem.
Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. The $V$ be length of time (in decades) until the first volcano eruption and $E$ the length of time (in decades) until the first earthquake. Compute the following:

- $\mathbb{E}[E^2]$ and $\text{Cov}[E, V]$. 
- The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
- The probability density function of $\min\{E, V\}$. 

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- The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
- The probability density function of $\min\{E, V\}$.
$E[E^2] = 2$ and $\text{Cov}[E, V] = 0$. 

Probability of no earthquake or eruption in first year is $e^{-2.3}$. Same for any year by the memoryless property. Expected number of quake/eruption-free years is $10e^{-2.3}$, which is approximately 7.44.
Poisson point processes — answers

- $E[E^2] = 2$ and $\text{Cov}[E, V] = 0$.
- Probability of no earthquake or eruption in first year is $e^{-\frac{2+1}{10}} = e^{-0.3}$ (see next part). Same for any year by memoryless property. Expected number of quake/eruption-free years is $10e^{-0.3} \approx 7.4$. 

18.440 Lecture 37
Poisson point processes — answers

- $E[E^2] = 2$ and $\text{Cov}[E, V] = 0$.
- Probability of no earthquake or eruption in first year is $e^{-(2+1)\frac{1}{10}} = e^{-\cdot3}$ (see next part). Same for any year by memoryless property. Expected number of quake/eruption-free years is $10e^{-\cdot3} \approx 7.4$.
- Probability density function of $\min\{E, V\}$ is $3e^{-(2+1)x}$ for $x \geq 0$, and 0 for $x < 0$. 