

**18.440: Lecture 35**

**Martingales and the optional stopping theorem**

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Martingales and stopping times

Optional stopping theorem

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## Martingale definition

- ▶ Let  $S$  be the probability space. Let  $X_0, X_1, X_2, \dots$  be a sequence of real random variables. Interpret  $X_i$  as price of asset at  $i$ th time step.

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- ▶ Say  $X_n$  sequence is a **martingale** if  $E[|X_n|] < \infty$  for all  $n$  and  $E[X_{n+1} | X_0, X_1, X_2, \dots, X_n] = X_n$  for all  $n$ .

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- ▶ If you are given a mathematical description of a process  $X_0, X_1, X_2, \dots$  then how can you check whether it is a martingale?
- ▶ Consider all of the information that you know after having seen  $X_0, X_1, \dots, X_n$ . Then try to figure out what additional (not yet known) randomness is involved in determining  $X_{n+1}$ . Use this to figure out the conditional expectation of  $X_{n+1}$ , and check to see whether this is always equal to the known  $X_n$  value.



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- ▶ Think of  $T$  as giving the time the asset will be sold if the price sequence is  $X_0, X_1, X_2, \dots$
- ▶ Say that  $T$  is a **stopping time** if the event that  $T = n$  depends only on the values  $X_i$  for  $i \leq n$ . In other words, the decision to sell at time  $n$  depends only on prices up to time  $n$ , not on (as yet unknown) future prices.

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- ▶ What if each  $A_i$  is  $1.01$  with probability  $.5$  and  $.99$  with probability  $.5$  and we write  $X_0 = 1$  and  $X_n = \prod_{i=1}^n A_i$  for  $n > 0$ ? Then is  $X_n$  a martingale?

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- ▶ Answer: yes.
- ▶ These are two classic martingale examples: a sum of independent random variables (each with mean zero) and a product of independent random variables (each with mean one).

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- ▶ Which of the following is a stopping time?
  1. The smallest  $T$  for which  $|X_T| = 50$
  2. The smallest  $T$  for which  $X_T \in \{-10, 100\}$
  3. The smallest  $T$  for which  $X_T = 0$ .
  4. The  $T$  at which the  $X_n$  sequence achieves the value  $17$  for the  $9$ th time.
  5. The value of  $T \in \{0, 1, 2, \dots, 100\}$  for which  $X_T$  is largest.
  6. The largest  $T \in \{0, 1, 2, \dots, 100\}$  for which  $X_T = 0$ .

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- ▶ Answer: first four, not last two.

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- ▶ Precisely, if you buy the asset at some time and adopt any strategy at all for deciding when to sell it, then the expected price at the time you sell is the price you originally paid.
- ▶ If market price is a martingale, you cannot make money in expectation by "timing the market."

## Doob's Optional Stopping Theorem: statement

- ▶ **Doob's Optional Stopping Theorem:** If the sequence  $X_0, X_1, X_2, \dots$  is a **bounded** martingale, and  $T$  is a stopping time, then the expected value of  $X_T$  is  $X_0$ .

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- ▶ Theorem can be proved by induction if *stopping time*  $T$  is bounded. Unbounded  $T$  requires a limit argument. (This is where boundedness of martingale is used.)

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- ▶ **Efficient market hypothesis**: new information is instantly absorbed into the stock value, so expected value of the stock tomorrow should be the value today. (If it were higher, statistical arbitrageurs would bid up today's price until this was not the case.)
- ▶ But what about interest, risk premium, etc.?
- ▶ According to the **fundamental theorem of asset pricing**, the discounted price  $\frac{X(n)}{A(n)}$ , where  $A$  is a risk-free asset, is a martingale with respect to **risk neutral probability**. More on this next lecture.

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- ▶ This means that the three-element sequence  $E[X], E[X|Y], X$  is a martingale.
- ▶ More generally if  $Y_i$  are any random variables, the sequence  $E[X], E[X|Y_1], E[X|Y_1, Y_2], E[X|Y_1, Y_2, Y_3], \dots$  is a martingale.

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- ▶ Call me!!! I love you! Alice 0

## More conditional probability martingale examples

- ▶ Example: let  $C$  be the amount of oil available for drilling under a particular piece of land. Suppose that ten geological tests are done that will ultimately determine the value of  $C$ . Let  $C_n$  be the **conditional expectation** of  $C$  *given* the outcome of the first  $n$  of these tests. Then the sequence  $C_0, C_1, C_2, \dots, C_{10} = C$  is a martingale.

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- ▶ Let  $A_i$  be my best guess at the probability that a basketball team will win the game, given the outcome of the first  $i$  minutes of the game. Then (assuming some “rationality” of my personal probabilities)  $A_i$  is a martingale.



## Are betting prices (as on Intrade) martingales?

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- ▶ **Factors that might deter smart professionals**: betting fees, low trading volumes, risk of betting-site insolvency, government regulations impeding money flow. Also amount of money required to sit in deposit, whether it gets interest.
- ▶ **Question**: In low-volume market, might market manipulators (bidding up prices to make their candidates look better) overwhelm statistical arbitrageurs? If so, more money available for arbitrageurs who hang around.

## Are betting prices (as on Intrade) martingales?

- ▶ **Roughly yes, if markets efficient**, which should be in high-volume markets (with low bid-ask spread). Otherwise smart professionals could make money in expectation by buying/selling. But such efforts correct prices. (Note: overall “need for money” doesn’t depend much outcome.)
- ▶ **Factors that might deter smart professionals**: betting fees, low trading volumes, risk of betting-site insolvency, government regulations impeding money flow. Also amount of money required to sit in deposit, whether it gets interest.
- ▶ **Question**: In low-volume market, might market manipulators (bidding up prices to make their candidates look better) overwhelm statistical arbitrageurs? If so, more money available for arbitrageurs who hang around.
- ▶ **Evidence for inefficiency**: price discrepancies, long shot bias.

# Examples

- ▶ Suppose that an asset price is a martingale that starts at 50 and changes by increments of  $\pm 1$  at each time step. What is the probability that the price goes down to 40 before it goes up to 70?

# Examples

- ▶ Suppose that an asset price is a martingale that starts at 50 and changes by increments of  $\pm 1$  at each time step. What is the probability that the price goes down to 40 before it goes up to 70?
- ▶ What is the probability that it goes down to 45 then up to 55 then down to 45 then up to 55 again — all before reaching either 0 or 100?