Outline

Markov chains

Examples

Ergodicity and stationarity
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Ergodicity and stationarity
Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.
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Interpret $X_n$ as state of the system at time $n$. 
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Interpret $X_n$ as state of the system at time $n$.

Sequence is called a **Markov chain** if we have a fixed collection of numbers $P_{ij}$ (one for each pair $i, j \in \{0, 1, \ldots, M\}$) such that whenever the system is in state $i$, there is probability $P_{ij}$ that system will next be in state $j$. 

Kind of an “almost memoryless” property. Probability distribution for next state depends only on the current state (and not on the rest of the state history).
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Precisely,

$$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P_{ij}.$$
Markov chains

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Simple example

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- If it’s rainy one day, there’s a 0.5 chance it will be rainy the next day, a 0.5 chance it will be sunny.
- If it’s sunny one day, there’s a 0.8 chance it will be sunny the next day, a 0.2 chance it will be rainy.
- In this climate, sun tends to last longer than rain.
- Given that it is rainy today, how many days do I expect to have to wait to see a sunny day?
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It is convenient to represent the collection of transition probabilities $P_{ij}$ as a matrix:

$$A = \begin{pmatrix}
P_{00} & P_{01} & \cdots & P_{0M} \\
P_{10} & P_{11} & \cdots & P_{1M} \\
\cdot & \cdot & \cdots & \cdot \\
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\end{pmatrix}$$

For this to make sense, we require $P_{ij} \geq 0$ for all $i, j$ and $\sum_{j=0}^{M} P_{ij} = 1$ for each $i$. That is, the rows sum to one.
Suppose that $p_i$ is the probability that system is in state $i$ at time zero.

What does the following product represent?

$$(p_0 \ p_1 \ \ldots \ \ p_M) \begin{bmatrix} P_{00} & P_{01} & \ldots & P_{0M} \\ P_{10} & P_{11} & \ldots & P_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ P_{M0} & P_{M1} & \ldots & P_{MM} \end{bmatrix}$$

Answer: the probability distribution at time one.

How about the following product?

$$(p_0 \ p_1 \ \ldots \ \ p_M) A^n$$

Answer: the probability distribution at time $n$. 

Transitions via matrices
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18.440 Lecture 33
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\end{array} \right)
$$

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18.440 Lecture 33
Powers of transition matrix

- We write $P_{ij}^{(n)}$ for the probability to go from state $i$ to state $j$ over $n$ steps.
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From the matrix point of view

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\]

- If $A$ is the one-step transition matrix, then $A^n$ is the $n$-step transition matrix.
Questions

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Answer: state sequence $X_i$ consists of i.i.d. random variables.
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What if each $P_{ij}$ is either one or zero?
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  - Answer: states never change.
- What if each $P_{ij}$ is either one or zero?
  - Answer: state evolution is deterministic.
Markov chains

Examples

Ergodicity and stationarity
Consider the simple weather example: If it’s rainy one day, there’s a .5 chance it will be rainy the next day, a .5 chance it will be sunny. If it’s sunny one day, there’s a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
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- Let rainy be state zero, sunny state one, and write the transition matrix by

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A = \begin{pmatrix}
.5 & .5 \\
.2 & .8
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Note that

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A^2 = \begin{pmatrix}
.64 & .35 \\
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Can compute \(A^{10}\) to be

\[
A^{10} = \begin{pmatrix}
.285719 & .714281 \\
.285713 & .714287
\end{pmatrix}
\]
Does relationship status have the Markov property?

- Single
- In a relationship
- It’s complicated
- Engaged
- Married

Can we assign a probability to each arrow?

Markov model implies time spent in any state (e.g., a marriage) before leaving is a geometric random variable.

Not true... Can we make a better model with more states?
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Ergodic Markov chains

- Say Markov chain is **ergodic** if some power of the transition matrix has all non-zero entries.
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Turns out that if chain has this property, then
\[ \pi_j := \lim_{n \to \infty} P_{ij}^{(n)} \]
exists and the \( \pi_j \) are the unique non-negative solutions of
\[ \pi_j = \sum_{k=0}^{M} \pi_k P_{kj} \]
that sum to one.
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This means that the row vector

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\pi = \begin{pmatrix} \pi_0 & \pi_1 & \ldots & \pi_M \end{pmatrix}
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is a left eigenvector of \( A \) with eigenvalue 1, i.e., \( \pi A = \pi \).
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We call \( \pi \) the *stationary distribution* of the Markov chain.
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Turns out that if chain has this property, then $\pi_j := \lim_{n \to \infty} P_{ij}^{(n)}$ exists and the $\pi_j$ are the unique non-negative solutions of $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ that sum to one.

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is a left eigenvector of $A$ with eigenvalue $1$, i.e., $\pi A = \pi$.

We call $\pi$ the *stationary distribution* of the Markov chain.

One can solve the system of linear equations $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ to compute the values $\pi_j$. Equivalent to considering $A$ fixed and solving $\pi A = \pi$. Or solving $(A - I)\pi = 0$. This determines $\pi$ up to a multiplicative constant, and fact that $\sum \pi_j = 1$ determines the constant.
Simple example

If $A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$, then we know

$$\pi A = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} = \pi.$$
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This means that $.5\pi_0 + .2\pi_1 = \pi_0$ and $.5\pi_0 + .8\pi_1 = \pi_1$ and we also know that $\pi_0 + \pi_1 = 1$. Solving these equations gives $\pi_0 = 2/7$ and $\pi_1 = 5/7$, so $\pi = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix}$. 

Indeed, $\pi A = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix} \begin{pmatrix} .5 \\ .2 \end{pmatrix} = \begin{pmatrix} 2/7 \ 5/7 \end{pmatrix} = \pi$. 

Recall that $A_{10} = \begin{pmatrix} .285719 \\ .714281 \end{pmatrix} \approx \begin{pmatrix} 2/7 & 5/7 \end{pmatrix}$. 

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Simple example

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Simple example

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- Recall that

$$A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix} \approx \begin{pmatrix} 2/7 & 5/7 \\ 2/7 & 5/7 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix}.$$