18.440: Lecture 33

Markov Chains

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Outline

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Markov chains

Examples

Ergodicity and stationarity

Markov chains

- Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.

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- Kind of an "almost memoryless" property. Probability distribution for next state depends only on the current state (and not on the rest of the state history).

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- Given that it is rainy today, how many days to I expect to have to wait to see a sunny day?
- Given that it is sunny today, how many days to I expect to have to wait to see a rainy day?
- Over the long haul, what fraction of days are sunny?

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- For this to make sense, we require $P_{ij} \geq 0$ for all i, j and $\sum_{j=0}^{M} P_{ij} = 1$ for each i. That is, the rows sum to one.

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- Answer: the probability distribution at time one.
- How about the following product?

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- Answer: the probability distribution at time n.

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- If A is the one-step transition matrix, then A^n is the n-step transition matrix.

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- Answer: state sequence X_i consists of i.i.d. random variables.
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- Answer: state evolution is deterministic.

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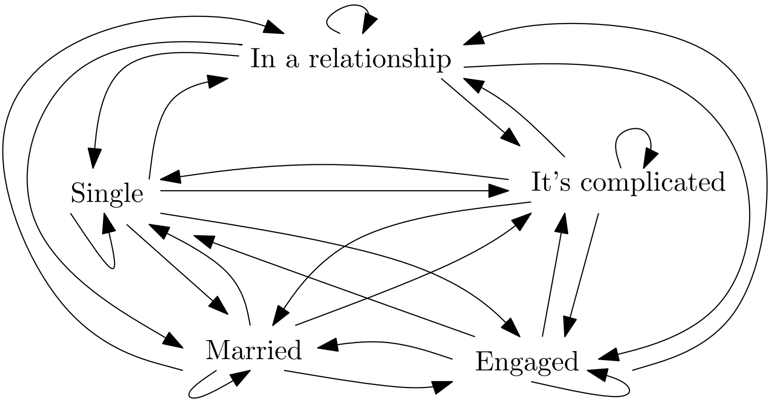
- Can compute $A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix}$

Does relationship status have the Markov property?



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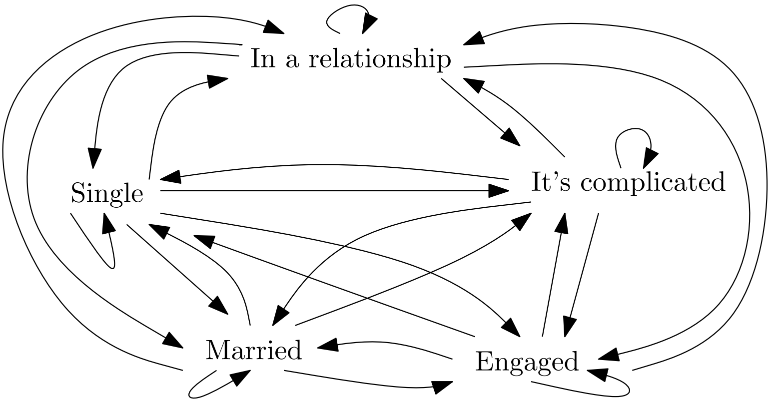

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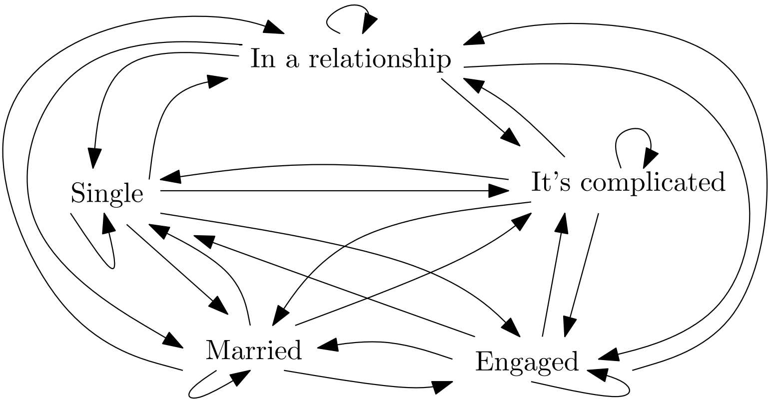

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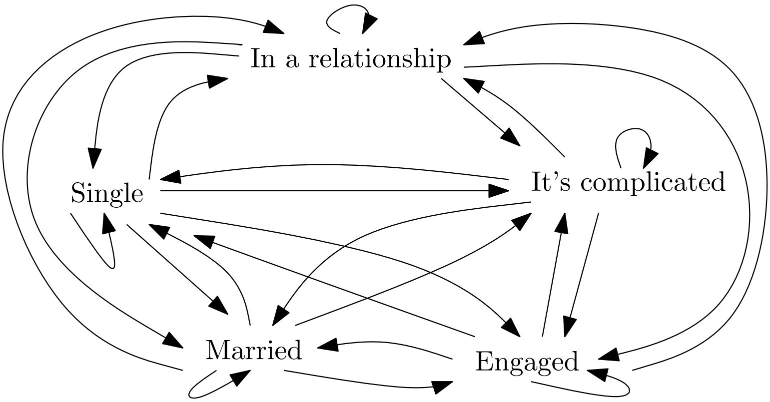

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- Not true... Can we make a better model with more states?

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- Turns out that if chain has this property, then $\pi_j := \lim_{n\to\infty} P_{ij}^{(n)}$ exists and the π_j are the unique non-negative solutions of $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ that sum to one.

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- One can solve the system of linear equations $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ to compute the values π_j. Equivalent to considering A fixed and solving $\pi A = \pi$. Or solving $(A - I)\pi = 0$. This determines π up to a multiplicative constant, and fact that $\sum \pi_j = 1$ determines the constant.

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- This means that $.5\pi_0 + .2\pi_1 = \pi_0$ and $.5\pi_0 + .8\pi_1 = \pi_1$ and we also know that $\pi_0 + \pi_1 = 1$. Solving these equations gives $\pi_0 = 2/7$ and $\pi_1 = 5/7$, so $\pi = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix}$.

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- Recall that
$A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix} \approx \begin{pmatrix} 2/7 & 5/7 \\ 2/7 & 5/7 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix}$