Outline

Central limit theorem

Proving the central limit theorem
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Recall: DeMoivre-Laplace limit theorem

- Let $X_i$ be an i.i.d. sequence of random variables. Write $S_n = \sum_{i=1}^{n} X_i$.

$S_n - np\sqrt{npq}$ describes "number of standard deviations that $S_n$ is above or below its mean".

Question: Does a similar statement hold if the $X_i$ are i.i.d. but have some other probability distribution?

Central limit theorem: Yes, if they have finite variance.
Recall: DeMoivre-Laplace limit theorem

- Let $X_i$ be an i.i.d. sequence of random variables. Write $S_n = \sum_{i=1}^{n} X_i$.
- Suppose each $X_i$ is 1 with probability $p$ and 0 with probability $q = 1 - p$.

DeMoivre-Laplace limit theorem:
\[
\lim_{n \to \infty} P\left\{ a \leq S_n - np \leq b \right\} \to \Phi(b) - \Phi(a).
\]
Here $\Phi(b) - \Phi(a) = P\{a \leq Z \leq b\}$ when $Z$ is a standard normal random variable.

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Central limit theorem: should be about $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$.  

18.440 Lecture 31
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Write \( B_n = \frac{X_1+X_2+\ldots+X_n-n\mu}{\sigma \sqrt{n}} \). Then \( B_n \) is the difference between \( S_n \) and its expectation, measured in standard deviation units.

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Recall: characteristic functions

Let $X$ be a random variable.

Characteristic functions are similar to moment generating functions in some ways. For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.

And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X''(0)$. Characteristic functions are well defined at all $t$ for all random variables $X$. 

Recall: characteristic functions

Let $X$ be a random variable.

The **characteristic function** of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.

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18.440 Lecture 31
Let $X$ be a random variable and $X_n$ a sequence of random variables.

Recall: the weak law of large numbers can be rephrased as the statement that $A_n = X_1 + X_2 + \ldots + X_n$ converges in law to $\mu$ (i.e., to the random variable that is equal to $\mu$ with probability one) as $n \to \infty$.

The central limit theorem can be rephrased as the statement that $B_n = X_1 + X_2 + \ldots + X_n - n\mu / \sigma \sqrt{n}$ converges in law to a standard normal random variable as $n \to \infty$. 

18.440 Lecture 31
Rephrasing the theorem

- Let $X$ be a random variable and $X_n$ a sequence of random variables.
- Say $X_n$ converge in distribution or converge in law to $X$ if
  \[ \lim_{n \to \infty} F_{X_n}(x) = F_X(x) \]
  at all $x \in \mathbb{R}$ at which $F_X$ is continuous.
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converges in law to a standard normal random variable as $n \to \infty$. 
Lévy’s continuity theorem (see Wikipedia): if

\[ \lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t) \]

for all \( t \), then \( X_n \) converge in law to \( X \).
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By this theorem, we can prove the central limit theorem by showing \( \lim_{n \to \infty} \phi_{B_n}(t) = e^{-t^2/2} \) for all \( t \).
Continuity theorems

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▶ **Moment generating function continuity theorem:** if moment generating functions \( M_{X_n}(t) \) are defined for all \( t \) and \( n \) and \( \lim_{n \to \infty} M_{X_n}(t) = M_X(t) \) for all \( t \), then \( X_n \) converge in law to \( X \).
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Proof of central limit theorem with moment generating functions

- Write \( Y = \frac{X - \mu}{\sigma} \). Then \( Y \) has mean zero and variance 1.
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- Write $Y = \frac{X-\mu}{\sigma}$. Then $Y$ has mean zero and variance 1.
- Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$. 

- Now $B_n$ is $\frac{1}{\sqrt{n}}$ times the sum of $n$ independent copies of $Y$. 
- So $M_{B_n}(t) = (M_Y(t/\sqrt{n}))^n = e^{ng(t/\sqrt{n})}$. 
- But $e^{ng(t/\sqrt{n})} \approx e^{n(g(t/\sqrt{n}))^2/2}$, in sense that LHS tends to $e^{t^2/2}$ as $n$ tends to infinity.
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- Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$.
- We know $g(0) = 0$. Also $M'_Y(0) = E[Y] = 0$ and $M''_Y(0) = E[Y^2] = \text{Var}[Y] = 1$. 

[18.440 Lecture 31]
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- We know $g(0) = 0$. Also $M_Y'(0) = E[Y] = 0$ and $M_Y''(0) = E[Y^2] = \text{Var}[Y] = 1$.
- Chain rule: $M_Y'(0) = g'(0)e^{g(0)} = g'(0) = 0$ and $M_Y''(0) = g''(0)e^{g(0)} + g'(0)^2e^{g(0)} = g''(0) = 1$. 

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- So \( g \) is a nice function with \( g(0) = g'(0) = 0 \) and \( g''(0) = 1 \).
  Taylor expansion: \( g(t) = \frac{t^2}{2} + o(t^2) \) for \( t \) near zero.
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▶ Write $Y = \frac{X-\mu}{\sigma}$. Then $Y$ has mean zero and variance 1.

▶ Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$.

▶ We know $g(0) = 0$. Also $M'_Y(0) = E[Y] = 0$ and $M''_Y(0) = E[Y^2] = \text{Var}[Y] = 1$.

▶ Chain rule: $M'_Y(0) = g'(0)e^{g(0)} = g'(0) = 0$ and $M''_Y(0) = g''(0)e^{g(0)} + g'(0)^2 e^{g(0)} = g''(0) = 1$.

▶ So $g$ is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = 1$. Taylor expansion: $g(t) = t^2/2 + o(t^2)$ for $t$ near zero.

▶ Now $B_n$ is $\frac{1}{\sqrt{n}}$ times the sum of $n$ independent copies of $Y$. 

18.440 Lecture 31
Proof of central limit theorem with moment generating functions

- Write $Y = \frac{X - \mu}{\sigma}$. Then $Y$ has mean zero and variance 1.
- Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$.
- We know $g(0) = 0$. Also $M'_Y(0) = E[Y] = 0$ and $M''_Y(0) = E[Y^2] = \text{Var}[Y] = 1$.
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- So $g$ is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = 1$. Taylor expansion: $g(t) = t^2/2 + o(t^2)$ for $t$ near zero.
- Now $B_n$ is $\frac{1}{\sqrt{n}}$ times the sum of $n$ independent copies of $Y$.
- So $M_{B_n}(t) = (M_Y(t/\sqrt{n}))^n = e^{ng(t/\sqrt{n})}$. 
Proof of central limit theorem with moment generating functions

- Write $Y = \frac{X-\mu}{\sigma}$. Then $Y$ has mean zero and variance 1.
- Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$.
- We know $g(0) = 0$. Also $M'_Y(0) = E[Y] = 0$ and $M''_Y(0) = E[Y^2] = \text{Var}[Y] = 1$.
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- But $e^{ng(\frac{t}{\sqrt{n}})} \approx e^{n(\frac{t}{\sqrt{n}})^2/2} = e^{t^2/2}$, in sense that LHS tends to $e^{t^2/2}$ as $n$ tends to infinity.
Proof of central limit theorem with characteristic functions

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- Moment generating function proof only applies if the moment generating function of \( X \) exists.
- But the proof can be repeated almost verbatim using characteristic functions instead of moment generating functions.
- Then it applies for any \( X \) with finite variance.
Write $\phi_Y(t) = E[e^{itY}]$ and $g(t) = \log \phi_Y(t)$. So $\phi_Y(t) = e^{g(t)}$.

We know $g(0) = 0$. Also $\phi_Y'(0) = iE[Y] = 0$ and $\phi_Y''(0) = i^2 E[Y^2] = -\text{Var}[Y] = -1$.

Chain rule: $\phi_Y'(0) = g'(0)e^{g(0)} = g'(0) = 0$ and $\phi_Y''(0) = g''(0)e^{g(0)} + g'(0)^2 e^{g(0)} = g''(0) = -1$.

So $g$ is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = -1$. Taylor expansion: $g(t) = -t^2/2 + o(t^2)$ for $t$ near zero.

Now $B_n$ is $1/\sqrt{n}$ times the sum of $n$ independent copies of $Y$.

So $\phi_{B_n}(t) = \left(\phi_Y(t/\sqrt{n})\right)^n = e^{ng(t/\sqrt{n})}$.

But $e^{ng(t/\sqrt{n})} \approx e^{-n(t/\sqrt{n})^2/2} = e^{-t^2/2}$, in sense that LHS tends to $e^{-t^2/2}$ as $n$ tends to infinity.
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By the chain rule, $\phi_Y'(0) = g'(0)e^{g(0)} = g'(0)$ and $\phi_Y''(0) = g''(0)e^{g(0)} + g'(0)^2 e^{g(0)} = g''(0)$. So $g$ is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = -1$. Taylor expansion: $g(t) = -t^2/2 + o(t^2)$ for $t$ near zero. 

Now $B_n$ is $\sqrt{n}$ times the sum of $n$ independent copies of $Y$. So $\phi_{B_n}(t) = \left(\phi_Y(t/\sqrt{n})\right)^n = e^{ng(t/\sqrt{n})}$. 

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18.440 Lecture 31
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Perspective

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- But, roughly speaking, if you have a lot of little random terms that are “mostly independent” — and no single term contributes more than a “small fraction” of the total sum — then the total sum should be “approximately” normal.

Example: if height is determined by lots of little mostly independent factors, then people’s heights should be normally distributed.

Not quite true... certain factors by themselves can cause a person to be a whole lot shorter or taller. Also, individual factors not really independent of each other.

Kind of true for homogenous population, ignoring outliers.
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