Outline

Weak law of large numbers: Markov/Chebyshev approach

Weak law of large numbers: characteristic function approach
Weak law of large numbers: Markov/Chebyshev approach

Weak law of large numbers: characteristic function approach
Markov’s and Chebyshev’s inequalities

- **Markov’s inequality**: Let $X$ be a random variable taking only non-negative values. Fix a constant $a > 0$. Then $P\{X \geq a\} \leq \frac{E[X]}{a}$. 

- **Chebyshev’s inequality**: If $X$ has finite mean $\mu$, variance $\sigma^2$, and $k > 0$ then $P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$. 

  - Proof: Note that $(X - \mu)^2$ is a non-negative random variable and $P\{|X - \mu| \geq k\} = P\{(X - \mu)^2 \geq k^2\}$. Now apply Markov’s inequality with $a = k^2$. 


Markov’s and Chebyshev’s inequalities

- **Markov’s inequality:** Let $X$ be a random variable taking only non-negative values. Fix a constant $a > 0$. Then $P\{X \geq a\} \leq \frac{E[X]}{a}$.

- **Proof:** Consider a random variable $Y$ defined by

  
  $Y = \begin{cases} 
  a & X \geq a \\
  0 & X < a 
  \end{cases}$.

  Since $X \geq Y$ with probability one, it follows that $E[X] \geq E[Y] = aP\{X \geq a\}$. Divide both sides by $a$ to get Markov’s inequality.

- **Chebyshev’s inequality:** If $X$ has finite mean $\mu$, variance $\sigma^2$, and $k > 0$ then $P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$.

- **Proof:** Note that $(X - \mu)^2$ is a non-negative random variable and $P\{|X - \mu| \geq k\} = P\{(X - \mu)^2 \geq k^2\}$. Now apply Markov’s inequality with $a = k^2$. 
Markov’s and Chebyshev’s inequalities

- **Markov’s inequality:** Let $X$ be a random variable taking only non-negative values. Fix a constant $a > 0$. Then $P\{X \geq a\} \leq \frac{E[X]}{a}$.

- **Proof:** Consider a random variable $Y$ defined by

  $Y = \begin{cases} a & X \geq a \\ 0 & X < a \end{cases}$. Since $X \geq Y$ with probability one, it follows that $E[X] \geq E[Y] = aP\{X \geq a\}$. Divide both sides by $a$ to get Markov’s inequality.

- **Chebyshev’s inequality:** If $X$ has finite mean $\mu$, variance $\sigma^2$, and $k > 0$ then

  $$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$
Markov’s and Chebyshev’s inequalities

- **Markov’s inequality:** Let $X$ be a random variable taking only non-negative values. Fix a constant $a > 0$. Then $\Pr\{X \geq a\} \leq \frac{E[X]}{a}$.

  **Proof:** Consider a random variable $Y$ defined by $Y = \begin{cases} a & X \geq a \\ 0 & X < a \end{cases}$. Since $X \geq Y$ with probability one, it follows that $E[X] \geq E[Y] = a \Pr\{X \geq a\}$. Divide both sides by $a$ to get Markov’s inequality.

- **Chebyshev’s inequality:** If $X$ has finite mean $\mu$, variance $\sigma^2$, and $k > 0$ then

  $$\Pr\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$ 

  **Proof:** Note that $(X - \mu)^2$ is a non-negative random variable and $\Pr\{|X - \mu| \geq k\} = \Pr\{(X - \mu)^2 \geq k^2\}$. Now apply Markov’s inequality with $a = k^2$. 

18.440 Lecture 30
Markov and Chebyshev: rough idea

- **Markov’s inequality:** Let \( X \) be a random variable taking only non-negative values with finite mean. Fix a constant \( a > 0 \). Then \( P\{X \geq a\} \leq \frac{E[X]}{a} \).

- **Chebyshev’s inequality:** If \( X \) has finite mean \( \mu \), variance \( \sigma^2 \), and \( k > 0 \) then \( P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2} \).

Inequalities allow us to deduce limited information about a distribution when we know only the mean (Markov) or the mean and variance (Chebyshev).
Markov and Chebyshev: rough idea

- **Markov’s inequality:** Let $X$ be a random variable taking only non-negative values with finite mean. Fix a constant $a > 0$. Then $P\{X \geq a\} \leq \frac{E[X]}{a}$.

- **Chebyshev’s inequality:** If $X$ has finite mean $\mu$, variance $\sigma^2$, and $k > 0$ then

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$
Markov and Chebyshev: rough idea

- **Markov’s inequality:** Let $X$ be a random variable taking only non-negative values with finite mean. Fix a constant $a > 0$. Then $P\{X \geq a\} \leq \frac{E[X]}{a}$.

- **Chebyshev’s inequality:** If $X$ has finite mean $\mu$, variance $\sigma^2$, and $k > 0$ then

  $$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$ 

- Inequalities allow us to deduce limited information about a distribution when we know only the mean (Markov) or the mean and variance (Chebyshev).
Markov and Chebyshev: rough idea

- **Markov's inequality**: Let $X$ be a random variable taking only non-negative values with finite mean. Fix a constant $a > 0$. Then $P\{X \geq a\} \leq \frac{E[X]}{a}$.

- **Chebyshev's inequality**: If $X$ has finite mean $\mu$, variance $\sigma^2$, and $k > 0$ then

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$  

- Inequalities allow us to deduce limited information about a distribution when we know only the mean (Markov) or the mean and variance (Chebyshev).

- **Markov**: if $E[X]$ is small, then it is not too likely that $X$ is large.
Markov and Chebyshev: rough idea

- **Markov’s inequality:** Let $X$ be a random variable taking only non-negative values with finite mean. Fix a constant $a > 0$. Then $P\{X \geq a\} \leq \frac{E[X]}{a}$.

- **Chebyshev’s inequality:** If $X$ has finite mean $\mu$, variance $\sigma^2$, and $k > 0$ then

  $$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$  

- Inequalities allow us to deduce limited information about a distribution when we know only the mean (Markov) or the mean and variance (Chebyshev).

- **Markov:** if $E[X]$ is small, then it is not too likely that $X$ is large.

- **Chebyshev:** if $\sigma^2 = \text{Var}[X]$ is small, then it is not too likely that $X$ is far from its mean.
Statement of weak law of large numbers

- Suppose $X_i$ are i.i.d. random variables with mean $\mu$. 
Suppose $X_i$ are i.i.d. random variables with mean $\mu$.

Then the value $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ is called the \textit{empirical average} of the first $n$ trials.
Suppose $X_i$ are i.i.d. random variables with mean $\mu$.

Then the value $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ is called the **empirical average** of the first $n$ trials.

We’d guess that when $n$ is large, $A_n$ is typically close to $\mu$. 

Indeed, weak law of large numbers states that for all $\epsilon > 0$ we have $\lim_{n \to \infty} P\{|A_n - \mu| > \epsilon\} = 0$. 

Example: as $n$ tends to infinity, the probability of seeing more than $\frac{1}{n}$ heads in $n$ fair coin tosses tends to zero.
Statement of weak law of large numbers

- Suppose $X_i$ are i.i.d. random variables with mean $\mu$.
- Then the value $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ is called the empirical average of the first $n$ trials.
- We’d guess that when $n$ is large, $A_n$ is typically close to $\mu$.
- Indeed, weak law of large numbers states that for all $\epsilon > 0$ we have $\lim_{n \to \infty} P\{|A_n - \mu| > \epsilon\} = 0$. 

Example: as $n$ tends to infinity, the probability of seeing more than 50001 $n$ heads in $n$ fair coin tosses tends to zero.
Suppose $X_i$ are i.i.d. random variables with mean $\mu$.

Then the value $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ is called the empirical average of the first $n$ trials.

We’d guess that when $n$ is large, $A_n$ is typically close to $\mu$.

Indeed, weak law of large numbers states that for all $\epsilon > 0$ we have $\lim_{n \to \infty} P\{|A_n - \mu| > \epsilon\} = 0$.

Example: as $n$ tends to infinity, the probability of seeing more than $.50001n$ heads in $n$ fair coin tosses tends to zero.
As above, let $X_i$ be i.i.d. random variables with mean $\mu$ and write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$.
Proof of weak law of large numbers in finite variance case

- As above, let $X_i$ be i.i.d. random variables with mean $\mu$ and write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$.
- By additivity of expectation, $\mathbb{E}[A_n] = \mu$.
Proof of weak law of large numbers in finite variance case

- As above, let $X_i$ be i.i.d. random variables with mean $\mu$ and write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$.
- By additivity of expectation, $\mathbb{E}[A_n] = \mu$.
- Similarly, $\text{Var}[A_n] = \frac{n\sigma^2}{n^2} = \sigma^2/n$. 

18.440 Lecture 30
As above, let $X_i$ be i.i.d. random variables with mean $\mu$ and write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$.

By additivity of expectation, $\mathbb{E}[A_n] = \mu$.

Similarly, $\text{Var}[A_n] = \frac{n \sigma^2}{n^2} = \frac{\sigma^2}{n}$.

By Chebyshev $P\{|A_n - \mu| \geq \epsilon\} \leq \frac{\text{Var}[A_n]}{\epsilon^2} = \frac{\sigma^2}{n \epsilon^2}$. 

No matter how small $\epsilon$ is, RHS will tend to zero as $n$ gets large.
Proof of weak law of large numbers in finite variance case

- As above, let \( X_i \) be i.i.d. random variables with mean \( \mu \) and write \( A_n := \frac{X_1 + X_2 + \cdots + X_n}{n} \).
- By additivity of expectation, \( \mathbb{E}[A_n] = \mu \).
- Similarly, \( \text{Var}[A_n] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \).
- By Chebyshev \( P\{|A_n - \mu| \geq \epsilon\} \leq \frac{\text{Var}[A_n]}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \).
- No matter how small \( \epsilon \) is, RHS will tend to zero as \( n \) gets large.
Outline

Weak law of large numbers: Markov/Chebyshev approach

Weak law of large numbers: characteristic function approach
Outline

Weak law of large numbers: Markov/Chebyshev approach

Weak law of large numbers: characteristic function approach
Question: does the weak law of large numbers apply no matter what the probability distribution for $X$ is?

Recall that in this strange case $A_n$ actually has the same probability distribution as $X$.

In particular, the $A_n$ are not tightly concentrated around any particular value even when $n$ is very large.

But in this case $E[|X|]$ was infinite. Does the weak law hold as long as $E[|X|]$ is finite, so that $\mu$ is well defined?

Yes. Can prove this using characteristic functions.
Question: does the weak law of large numbers apply no matter what the probability distribution for $X$ is?

Is it always the case that if we define $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ then $A_n$ is typically close to some fixed value when $n$ is large?

What if $X$ is Cauchy?

Recall that in this strange case $A_n$ actually has the same probability distribution as $X$.

In particular, the $A_n$ are not tightly concentrated around any particular value even when $n$ is very large.

But in this case $E[|X|]$ was infinite. Does the weak law hold as long as $E[|X|]$ is finite, so that $\mu$ is well defined?

Yes. Can prove this using characteristic functions.
Question: does the weak law of large numbers apply no matter what the probability distribution for $X$ is?

Is it always the case that if we define $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ then $A_n$ is typically close to some fixed value when $n$ is large?

What if $X$ is Cauchy?
Question: does the weak law of large numbers apply no matter what the probability distribution for \( X \) is?

Is it always the case that if we define \( A_n := \frac{X_1+X_2+...+X_n}{n} \) then \( A_n \) is typically close to some fixed value when \( n \) is large?

What if \( X \) is Cauchy?

Recall that in this strange case \( A_n \) actually has the same probability distribution as \( X \).
Question: does the weak law of large numbers apply no matter what the probability distribution for $X$ is?

Is it always the case that if we define $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ then $A_n$ is typically close to some fixed value when $n$ is large?

What if $X$ is Cauchy?

Recall that in this strange case $A_n$ actually has the same probability distribution as $X$.

In particular, the $A_n$ are not tightly concentrated around any particular value even when $n$ is very large.
Question: does the weak law of large numbers apply no matter what the probability distribution for $X$ is?

Is it always the case that if we define $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ then $A_n$ is typically close to some fixed value when $n$ is large?

What if $X$ is Cauchy?

Recall that in this strange case $A_n$ actually has the same probability distribution as $X$.

In particular, the $A_n$ are not tightly concentrated around any particular value even when $n$ is very large.

But in this case $E[|X|]$ was infinite. Does the weak law hold as long as $E[|X|]$ is finite, so that $\mu$ is well defined?
Question: does the weak law of large numbers apply no matter what the probability distribution for $X$ is?

Is it always the case that if we define $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$ then $A_n$ is typically close to some fixed value when $n$ is large?

What if $X$ is Cauchy?

Recall that in this strange case $A_n$ actually has the same probability distribution as $X$.

In particular, the $A_n$ are not tightly concentrated around any particular value even when $n$ is very large.

But in this case $E[|X|]$ was infinite. Does the weak law hold as long as $E[|X|]$ is finite, so that $\mu$ is well defined?

Yes. Can prove this using characteristic functions.
Let $X$ be a random variable.
Let $X$ be a random variable.

The **characteristic function** of $X$ is defined by

$$\phi(t) = \phi_X(t) := E[e^{itX}].$$

Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.

Characteristic functions are similar to moment generating functions in some ways. For example, $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$, just as $M_{X+Y}(t) = M_X(t)M_Y(t)$, if $X$ and $Y$ are independent.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.

And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.

But characteristic functions have an advantage: they are well defined at all $t$ for all random variables $X$. 
Let $X$ be a random variable.

The **characteristic function** of $X$ is defined by 

$$\phi(t) = \phi_X(t) := E[e^{itX}]$$. Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i \sin(t)$.

Characteristic functions are similar to moment generating functions in some ways.

For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.

And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.

But characteristic functions have an advantage: they are well defined at all $t$ for all random variables $X$. 

18.440 Lecture 30
Let $X$ be a random variable.

The **characteristic function** of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.

Characteristic functions are similar to moment generating functions in some ways.

But characteristic functions have an advantage: they are well defined at all $t$ for all random variables $X$. 

---

18.440 Lecture 30
Characteristics functions

Let $X$ be a random variable.

The **characteristic function** of $X$ is defined by

$$\phi(t) = \phi_X(t) := E[e^{itX}]$$. Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i \sin(t)$.

Characteristic functions are similar to moment generating functions in some ways.

For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.
Let $X$ be a random variable.

The **characteristic function** of $X$ is defined by

$$
\phi(t) = \phi_X(t) := E[e^{itX}].
$$

Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.

Characteristic functions are similar to moment generating functions in some ways.

For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$. 

Let $X$ be a random variable.

The **characteristic function** of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.

Characteristic functions are similar to moment generating functions in some ways.

For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.

And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$. 
Let $X$ be a random variable.

The characteristic function of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.

Characteristic functions are similar to moment generating functions in some ways.

For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.

And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.

But characteristic functions have an advantage: they are well defined at all $t$ for all random variables $X$. 
Continuity theorems

- Let $X$ be a random variable and $X_n$ a sequence of random variables.

- Say $X_n$ converge in distribution or converge in law to $X$ if
\[ \lim_{n \to \infty} F_{X_n}(x) = F_X(x) \]
at all $x \in \mathbb{R}$ at which $F_X$ is continuous.

- The weak law of large numbers can be rephrased as the statement that
\[ A_n \] converge in law to $\mu$ (i.e., to the random variable that is equal to $\mu$ with probability one).

- Lévy's continuity theorem (see Wikipedia): if
\[ \lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t) \]
for all $t$, then $X_n$ converge in law to $X$.

- By this theorem, we can prove the weak law of large numbers by showing
\[ \lim_{n \to \infty} \phi_{A_n}(t) = \phi_{\mu}(t) = e^{it\mu} \]
for all $t$. In the special case that $\mu = 0$, this amounts to showing
\[ \lim_{n \to \infty} \phi_{A_n}(t) = 1 \] for all $t$. 

18.440 Lecture 30
Continuity theorems

- Let $X$ be a random variable and $X_n$ a sequence of random variables.
- Say $X_n$ converge in distribution or converge in law to $X$ if
  \[ \lim_{n \to \infty} F_{X_n}(x) = F_X(x) \]
  at all $x \in \mathbb{R}$ at which $F_X$ is continuous.

- The weak law of large numbers can be rephrased as the statement that
  \[ A_n \text{ converges in law to } \mu \]
  (i.e., to the random variable that is equal to $\mu$ with probability one).

- Lévy's continuity theorem (see Wikipedia):
  \[ \lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t) \]
  for all $t$, then $X_n$ converge in law to $X$.

- By this theorem, we can prove the weak law of large numbers by showing
  \[ \lim_{n \to \infty} \phi_{A_n}(t) = \phi_{\mu}(t) = e^{it\mu} \]
  for all $t$. In the special case that $\mu = 0$, this amounts to showing
  \[ \lim_{n \to \infty} \phi_{A_n}(t) = 1 \]
  for all $t$. In the special case that $\mu = 0$, this amounts to showing
Let $X$ be a random variable and $X_n$ a sequence of random variables.

Say $X_n$ converge in distribution or converge in law to $X$ if $\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$ at all $x \in \mathbb{R}$ at which $F_X$ is continuous.

The weak law of large numbers can be rephrased as the statement that $A_n$ converges in law to $\mu$ (i.e., to the random variable that is equal to $\mu$ with probability one).

Lévy’s continuity theorem (see Wikipedia): if $\lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t)$ for all $t$, then $X_n$ converge in law to $X$. By this theorem, we can prove the weak law of large numbers by showing $\lim_{n \to \infty} \phi_{A_n}(t) = \phi_\mu(t) = e^{it\mu}$ for all $t$. In the special case that $\mu = 0$, this amounts to showing $\lim_{n \to \infty} \phi_{A_n}(t) = 1$ for all $t$. 

18.440 Lecture 30
Continuity theorems

- Let $X$ be a random variable and $X_n$ a sequence of random variables.
- Say $X_n$ converge in distribution or converge in law to $X$ if \( \lim_{n \to \infty} F_{X_n}(x) = F_X(x) \) at all \( x \in \mathbb{R} \) at which \( F_X \) is continuous.
- The weak law of large numbers can be rephrased as the statement that $A_n$ converges in law to $\mu$ (i.e., to the random variable that is equal to $\mu$ with probability one).
- Lévy’s continuity theorem (see Wikipedia): if

\[
\lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t)
\]

for all $t$, then $X_n$ converge in law to $X$. 
Continuity theorems

- Let $X$ be a random variable and $X_n$ a sequence of random variables.
- Say $X_n$ converge in distribution or converge in law to $X$ if $\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$ at all $x \in \mathbb{R}$ at which $F_X$ is continuous.
- The weak law of large numbers can be rephrased as the statement that $A_n$ converges in law to $\mu$ (i.e., to the random variable that is equal to $\mu$ with probability one).
- Lévy’s continuity theorem (see Wikipedia): if
  \[
  \lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t)
  \]
  for all $t$, then $X_n$ converge in law to $X$.
- By this theorem, we can prove the weak law of large numbers by showing $\lim_{n \to \infty} \phi_{A_n}(t) = \phi_{\mu}(t) = e^{it\mu}$ for all $t$. In the special case that $\mu = 0$, this amounts to showing $\lim_{n \to \infty} \phi_{A_n}(t) = 1$ for all $t$. 

18.440 Lecture 30
Proof of weak law of large numbers in finite mean case

As above, let $X_i$ be i.i.d. instances of random variable $X$ with mean zero. Write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$. Weak law of large numbers holds for i.i.d. instances of $X$ if and only if it holds for i.i.d. instances of $X - \mu$. Thus it suffices to prove the weak law in the mean zero case.
Proof of weak law of large numbers in finite mean case

- As above, let $X_i$ be i.i.d. instances of random variable $X$ with mean zero. Write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$. Weak law of large numbers holds for i.i.d. instances of $X$ if and only if it holds for i.i.d. instances of $X - \mu$. Thus it suffices to prove the weak law in the mean zero case.

- Consider the characteristic function $\phi_X(t) = E[e^{itX}]$. 

\[\phi_X(t) = E[e^{it\frac{X_1 + X_2 + \ldots + X_n}{n}}] = e^{ng\left(\frac{t}{n}\right)}\]
Proof of weak law of large numbers in finite mean case

As above, let $X_i$ be i.i.d. instances of random variable $X$ with mean zero. Write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$. Weak law of large numbers holds for i.i.d. instances of $X$ if and only if it holds for i.i.d. instances of $X - \mu$. Thus it suffices to prove the weak law in the mean zero case.

Consider the characteristic function $\phi_X(t) = E[e^{itX}]$.

Since $E[X] = 0$, we have $\phi'_X(0) = E[\frac{\partial}{\partial t} e^{itX}]_{t=0} = iE[X] = 0$. 

By Lévy's continuity theorem, the $A_n$ converge in law to 0 (i.e., to the random variable that is 0 with probability one).
Proof of weak law of large numbers in finite mean case

- As above, let $X_i$ be i.i.d. instances of random variable $X$ with mean zero. Write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$. Weak law of large numbers holds for i.i.d. instances of $X$ if and only if it holds for i.i.d. instances of $X - \mu$. Thus it suffices to prove the weak law in the mean zero case.

- Consider the characteristic function $\phi_X(t) = E[e^{itX}]$.

- Since $E[X] = 0$, we have $\phi_X'(0) = E[\frac{\partial}{\partial t} e^{itX}]_{t=0} = iE[X] = 0$.

- Write $g(t) = \log \phi_X(t)$ so $\phi_X(t) = e^{g(t)}$. Then $g(0) = 0$ and (by chain rule) $g'(0) = \lim_{\epsilon \to 0} \frac{g(\epsilon) - g(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{g(\epsilon)}{\epsilon} = 0$. 
Proof of weak law of large numbers in finite mean case

As above, let $X_i$ be i.i.d. instances of random variable $X$ with mean zero. Write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$. Weak law of large numbers holds for i.i.d. instances of $X$ if and only if it holds for i.i.d. instances of $X - \mu$. Thus it suffices to prove the weak law in the mean zero case.

Consider the characteristic function $\phi_X(t) = E[e^{itX}]$.

Since $E[X] = 0$, we have $\phi'_X(0) = E[\frac{\partial}{\partial t} e^{itX}]_{t=0} = iE[X] = 0$.

Write $g(t) = \log \phi_X(t)$ so $\phi_X(t) = e^{g(t)}$. Then $g(0) = 0$ and (by chain rule) $g'(0) = \lim_{\epsilon \to 0} \frac{g(\epsilon) - g(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\frac{g(\epsilon)}{\epsilon}}{\epsilon} = 0$.

Now $\phi_{A_n}(t) = \phi_X(t/n)^n = e^{ng(t/n)}$. Since $g(0) = g'(0) = 0$, we have $\lim_{n \to \infty} ng(t/n) = \lim_{n \to \infty} t \frac{g(t/n)}{t} = 0$ if $t$ is fixed. Thus $\lim_{n \to \infty} e^{ng(t/n)} = 1$ for all $t$. By Lévy's continuity theorem, the $A_n$ converge in law to 0 (i.e., to the random variable that is 0 with probability one).
As above, let $X_i$ be i.i.d. instances of random variable $X$ with mean zero. Write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$. Weak law of large numbers holds for i.i.d. instances of $X$ if and only if it holds for i.i.d. instances of $X - \mu$. Thus it suffices to prove the weak law in the mean zero case.

Consider the characteristic function $\phi_X(t) = E[e^{itX}]$.

Since $E[X] = 0$, we have $\phi'_X(0) = E[\frac{\partial}{\partial t} e^{itX}]_{t=0} = iE[X] = 0$.

Write $g(t) = \log \phi_X(t)$ so $\phi_X(t) = e^{g(t)}$. Then $g(0) = 0$ and (by chain rule) $g'(0) = \lim_{\epsilon \to 0} \frac{g(\epsilon) - g(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{g(\epsilon)}{\epsilon} = 0$.

Now $\phi_{A_n}(t) = \phi_X(t/n)^n = e^{ng(t/n)}$. Since $g(0) = g'(0) = 0$ we have $\lim_{n \to \infty} ng(t/n) = \lim_{n \to \infty} t \frac{g(t/n)}{t/n} = 0$ if $t$ is fixed. Thus $\lim_{n \to \infty} e^{ng(t/n)} = 1$ for all $t$. 

By Lévy's continuity theorem, the $A_n$ converge in law to 0 (i.e., to the random variable that is 0 with probability one).
Proof of weak law of large numbers in finite mean case

As above, let $X_i$ be i.i.d. instances of random variable $X$ with mean zero. Write $A_n := \frac{X_1 + X_2 + \ldots + X_n}{n}$. Weak law of large numbers holds for i.i.d. instances of $X$ if and only if it holds for i.i.d. instances of $X - \mu$. Thus it suffices to prove the weak law in the mean zero case.

Consider the characteristic function $\phi_X(t) = E[e^{itX}]$.

Since $E[X] = 0$, we have $\phi_X'(0) = E\left[\frac{\partial}{\partial t} e^{itX}\right]_{t=0} = iE[X] = 0$.

Write $g(t) = \log \phi_X(t)$ so $\phi_X(t) = e^{g(t)}$. Then $g(0) = 0$ and (by chain rule) $g'(0) = \lim_{\epsilon \to 0} \frac{g(\epsilon) - g(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{g(\epsilon)}{\epsilon} = 0$.

Now $\phi_{A_n}(t) = \phi_X(t/n)^n = e^{ng(t/n)}$. Since $g(0) = g'(0) = 0$ we have $\lim_{n \to \infty} ng(t/n) = \lim_{n \to \infty} t \frac{g(t/n)}{n} = 0$ if $t$ is fixed. Thus $\lim_{n \to \infty} e^{ng(t/n)} = 1$ for all $t$.

By Lévy’s continuity theorem, the $A_n$ converge in law to 0 (i.e., to the random variable that is 0 with probability one).