18.440: Lecture 30

Weak law of large numbers

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Outline

Weak law of large numbers: Markov/Chebyshev approach

Weak law of large numbers: characteristic function approach

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Markov's and Chebyshev's inequalities

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$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$

- **Proof:** Note that $(X - \mu)^2$ is a non-negative random variable and $P\{|X - \mu| \geq k\} = P\{(X - \mu)^2 \geq k^2\}$. Now apply Markov's inequality with $a = k^2$.

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- **Chebyshev:** if $\sigma^2 = \mathrm{Var}[X]$ is small, then it is not too likely that X is far from its mean.

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- Indeed, **weak law of large numbers** states that for all $\epsilon > 0$ we have $\lim_{n\to\infty} P\{|A_n - \mu| > \epsilon\} = 0$.
- Example: as n tends to infinity, the probability of seeing more than $.50001n$ heads in n fair coin tosses tends to zero.

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- No matter how small ϵ is, RHS will tend to zero as n gets large.

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- Yes. Can prove this using characteristic functions.

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- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- But characteristic functions have an advantage: they are well defined at all t for all random variables X.

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  for all t, then X_n converge in law to X.
- By this theorem, we can prove the weak law of large numbers by showing $\lim_{n\to\infty} \phi_{A_n}(t) = \phi_\mu(t) = e^{it\mu}$ for all t. In the special case that $\mu = 0$, this amounts to showing $\lim_{n\to\infty} \phi_{A_n}(t) = 1$ for all t.

Proof of weak law of large numbers in finite mean case

- As above, let X_i be i.i.d. instances of random variable X with mean zero. Write $A_n := \frac{X_1+X_2+\ldots+X_n}{n}$. Weak law of large numbers holds for i.i.d. instances of X if and only if it holds for i.i.d. instances of $X - \mu$. Thus it suffices to prove the weak law in the mean zero case.

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- Write $g(t) = \log \phi_X(t)$ so $\phi_X(t) = e^{g(t)}$. Then $g(0)=0$ and (by chain rule) $g'(0) = \lim_{\epsilon\to 0}\frac{g(\epsilon)-g(0)}{\epsilon} = \lim_{\epsilon\to 0}\frac{g(\epsilon)}{\epsilon} = 0$.

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