18.440: Lecture 27

Moment generating functions and characteristic functions

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Outline

Moment generating functions

Characteristic functions

Continuity theorems and perspective

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Continuity theorems and perspective

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- If $b > 0$ and $t > 0$ then $E[e^{tX}] \geq E[e^{t \min\{X,b\}}] \geq P\{X \geq b\} e^{tb}$.
- If X takes both positive and negative values with positive probability then $M(t)$ grows at least exponentially fast in $|t|$ as $|t| \to \infty$.

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- Another way to think of this: write $e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots$.
- Taking expectations gives $E[e^{tX}] = 1 + tm_1 + \frac{t^2m_2}{2!} + \frac{t^3m_3}{3!} + \dots$, where m_k is the kth moment. The kth derivative at zero is m_k.

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- By independence, $M_Z(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$ for all t.

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- In other words, adding independent random variables corresponds to multiplying moment generating functions.

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- This a big reason for studying moment generating functions. It helps us understand what happens when we sum up a lot of independent copies of the same random variable.

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- Answer: Yes. $M_Z(t) = E[e^{tZ}] = E[e^{tX+bt}] = e^{bt}M_X(t)$.
- Latter answer is the special case of $M_Z(t) = M_X(t)M_Y(t)$ where Y is the constant random variable b.

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- Answer: $M_X(t) = E[e^{tx}] = \sum_{n=0}^{\infty} \frac{e^{tn}e^{-\lambda}\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{-\lambda}e^{\lambda e^t} = \exp[\lambda(e^t - 1)]$.

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- We know that if you add independent Poisson random variables with parameters λ_1 and λ_2 you get a Poisson random variable of parameter $\lambda_1 + \lambda_2$. How is this fact manifested in the moment generating function?

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- Answer: Z has same law as $\sigma X + \mu$, so $M_Z(t) = M(\sigma t) e^{\mu t} = \exp\{\frac{\sigma^2 t^2}{2} + \mu t\}$.

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- Exponential calculation above works for $t < \lambda$. What happens when $t > \lambda$? Or as t approaches λ from below?
- $M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda - t)x} dx = \infty$ if $t \geq \lambda$.

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- Answer: $M_X(0) = 1$ (as is true for any X) but otherwise $M_X(t)$ is infinite for all $t \neq 0$.
- Informal statement: moment generating functions are not defined for distributions with fat tails.

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- And if X has an mth moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- But characteristic functions have a distinct advantage: they are always well defined for all t even if f_X decays slowly.

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Perspective

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- Proofs using characteristic functions apply in more generality, but they require you to remember how to exponentiate imaginary numbers.

Perspective

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- Characteristic functions are *Fourier transforms* of the corresponding distribution density functions and encode "periodicity" patterns. For example, if X is integer valued, $\phi_X(t) = E[e^{itX}]$ will be 1 whenever t is a multiple of 2π.

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- **Moment generating analog:** if moment generating functions $M_{X_n}(t)$ are defined for all t and n and $\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$ for all t, then X_n converge in law to X.