18.440: Lecture 23

Sums of independent random variables

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- Differentiating both sides gives $f_{X+Y}(a) = \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy.$

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- Latter formula makes some intuitive sense. We're integrating over the set of x, y pairs that add up to a.

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- Worth memorizing.

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- $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy = \int_0^1 f_X(a-y)$ which is the length of $[0,1] \cap [a-1,a]$.

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- That's a when $a \in [0,1]$ and $2-a$ when $a \in [1,2]$ and 0 otherwise.

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- We can interpret Z as time slot where nth head occurs in i.i.d. sequence of p-coin tosses.
- So Z is negative binomial (n, p). So $P\{Z = k\} = \binom{k-1}{n-1}p^{n-1}(1-p)^{k-n}p$.

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- By induction, would suffice to show that a gamma $(\lambda, 1)$ plus an independent gamma (λ, n) is a gamma $(\lambda, n+1)$.

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- Up to an a-independent multiplicative constant, this is

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- Letting $x = y/a$, this becomes $e^{-\lambda a} a^{s+t-1} \int_0^1 (1-x)^{s-1} x^{t-1} dx$.
- This is (up to multiplicative constant) $e^{-\lambda a} a^{s+t-1}$. Constant must be such that integral from $-\infty$ to ∞ is 1. Conclude that $X+Y$ is gamma $(\lambda, s+t)$.

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- Generally: if independent random variables X_j are normal (μ_j, σ_j^2) then $\sum_{j=1}^n X_j$ is normal $(\sum_{j=1}^n \mu_j, \sum_{j=1}^n \sigma_j^2)$.

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- Yes, Poisson $\lambda_1 + \lambda_2$. Can be seen from Poisson point process interpretation.