18.440: Lecture 20

Exponential random variables

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MIT
Outline

Exponential random variables

Minimum of independent exponentials

Memoryless property

Relationship to Poisson random variables
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Relationship to Poisson random variables
Say $X$ is an **exponential random variable of parameter** $\lambda$
when its probability distribution function is

$$f(x) = \begin{cases} 
\lambda e^{-\lambda x} & x \geq 0 \\
0 & x < 0 
\end{cases}.$$
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For $a > 0$ have

$$F_X(a) = \int_0^a f(x)\,dx = \int_0^a \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\bigg|_0^a = 1 - e^{-\lambda a}.$$
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Thus \( P\{X < a\} = 1 - e^{-\lambda a} \) and \( P\{X > a\} = e^{-\lambda a} \).
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Thus \( P\{X < a\} = 1 - e^{-\lambda a} \) and \( P\{X > a\} = e^{-\lambda a} \).

Formula \( P\{X > a\} = e^{-\lambda a} \) is very important in practice.
Suppose $X$ is exponential with parameter $\lambda$, so $f_X(x) = \lambda e^{-\lambda x}$ when $x \geq 0$. 

Integration by parts gives $E[X^n] = -\int_0^\infty nx^{n-1}\lambda e^{-\lambda x} dx + x^n\lambda e^{-\lambda x} \bigg|_0^\infty$.

We get $E[X^n] = n\lambda E[X^{n-1}]$.

$E[X^0] = E[1] = 1,$
$E[X] = 1/\lambda,$
$E[X^2] = 2/\lambda^2,$
$E[X^n] = n!\lambda^n$.

If $\lambda = 1$, the $E[X^n] = n!$. Could take this as definition of $n!$. It makes sense for $n = 0$ and for non-integer $n$. 

Variance: $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/\lambda^2$. 

Moment formula
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CLAIM: If \( X_1 \) and \( X_2 \) are independent and exponential with parameters \( \lambda_1 \) and \( \lambda_2 \) then \( X = \min\{X_1, X_2\} \) is exponential with parameter \( \lambda = \lambda_1 + \lambda_2 \).
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How could we prove this?
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How could we prove this?

Have various ways to describe random variable $Y$: via density function $f_Y(x)$, or cumulative distribution function $F_Y(a) = P\{Y \leq a\}$, or function $P\{Y > a\} = 1 - F_Y(a)$.
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- Last one has simple form for exponential random variables. We have $P\{Y > a\} = e^{-\lambda a}$ for $a \in [0, \infty)$.
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Note: \( X > a \) if and only if \( X_1 > a \) and \( X_2 > a \).
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If $X_1, \ldots, X_n$ are independent exponential with $\lambda_1, \ldots \lambda_n$, then
\[
\min\{X_1, \ldots X_n\} \text{ is exponential with } \lambda = \lambda_1 + \ldots + \lambda_n.
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- **Memoryless property:** If $X$ represents the time until an event occurs, then *given* that we have seen no event up to time $b$, the conditional distribution of the remaining time till the event is the same as it originally was.
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- To make this precise, we ask what is the probability distribution of $Y = X - b$ *conditioned on* $X > b$?
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To make this precise, we ask what is the probability distribution of $Y = X - b$ *conditioned on* $X > b$?

We can characterize the conditional law of $Y$, given $X > b$, by computing $P(Y > a|X > b)$ for each $a$. 

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- That is, we compute
  \[ P(X - b > a | X > b) = P(X > b + a | X > b). \]
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- By definition of conditional probability, this is just
  \[ P\{X > b + a\}/P\{X > b\} = e^{--\lambda(b+a)}/e^{--\lambda b} = e^{--\lambda a}. \]
- Thus, conditional law of $X - b$ *given* that $X > b$ is same as the original law of $X$. 
Similar property holds for geometric random variables.
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If we plan to toss a coin until the first heads comes up, then we have a .5 chance to get a heads in one step, a .25 chance in two steps, etc.
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Given that the first 5 tosses are all tails, there is conditionally a .5 chance we get our first heads on the 6th toss, a .25 chance on the 7th toss, etc.
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Despite our having had five tails in a row, our expectation of the amount of time remaining until we see a heads is the same as it originally was.
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Alice: Still fifty fifty.
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Alice: It’s a math puzzle. You always assume a normal coin.

Bob: No, that’s your mistake. You should never assume that, because maybe somebody tampered with the coin.
Exchange overheard on a Logan airport shuttle

▶ Alice: Yeah, yeah, I get it. I can’t win here.

▶ Bob: No, I don’t think you get it yet. It’s a subtle point in statistics. It’s very important.

Exchange continued for duration of shuttle ride (Alice increasingly irritated, Bob increasingly patronizing).

▶ Raises interesting question about memoryless property.

▶ Suppose the duration of a couple’s relationship is exponential with $\lambda - 1$ equal to two weeks.

▶ Given that it has lasted for 10 weeks so far, what is the conditional probability that it will last an additional week?

▶ How about an additional four weeks? Ten weeks?
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How about an additional four weeks? Ten weeks?
Alice assumes Bob means “independent tosses of a fair coin.” Under this assumption, all $2^{11}$ outcomes of eleven-coin-toss sequence are equally likely. Bob considers HHHHHHHHHHH more likely than HHHHHHHHHHT, since former could result from a faulty coin.
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Alice sees Bob’s point but considers it annoying and churlish to ask about coin toss sequence and criticize listener for assuming this means “independent tosses of fair coin”.
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Without that assumption, Alice has no idea what context Bob has in mind. (An environment where two-headed novelty coins are common? Among coin-tossing cheaters with particular agendas?...)
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Alice: you need assumptions to convert stories into math.
Remark on Alice and Bob

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- Alice sees Bob’s point but considers it annoying and churlish to ask about coin toss sequence and criticize listener for assuming this means “independent tosses of fair coin”.

- Without that assumption, Alice has no idea what context Bob has in mind. (An environment where two-headed novelty coins are common? Among coin-tossing cheaters with particular agendas?...)

- Alice: you need assumptions to convert stories into math.

- Bob: good to question assumptions.
Suppose you start at time zero with $n$ radioactive particles. Suppose that each one (independently of the others) will decay at a random time, which is an exponential random variable with parameter $\lambda$. 

Let $T$ be the amount of time until no particles are left. What are $E[T]$ and $\text{Var}[T]$?

Let $T_1$ be the amount of time you wait until the first particle decays, $T_2$ the amount of additional time until the second particle decays, etc., so that $T = T_1 + T_2 + \ldots + T_n$.

Claim: $T_1$ is exponential with parameter $n\lambda$.

Claim: $T_2$ is exponential with parameter $(n-1)\lambda$.

And so forth.

$E[T] = \sum_{i=1}^{n} E[T_i] = \lambda - \frac{1}{\lambda} \sum_{j=1}^{n} \frac{1}{j}$ and (by independence) $\text{Var}[T] = \sum_{i=1}^{n} \text{Var}[T_i] = \frac{1}{\lambda^2} - \frac{2}{\lambda^2} \sum_{j=1}^{n} \frac{1}{j^2}$. 

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Claim: \( T_2 \) is exponential with parameter \( (n - 1)\lambda \).
Suppose you start at time zero with \( n \) radioactive particles. Suppose that each one (independently of the others) will decay at a random time, which is an exponential random variable with parameter \( \lambda \).

Let \( T \) be amount of time until no particles are left. What are \( E[T] \) and \( \text{Var}[T] \)?

Let \( T_1 \) be the amount of time you wait until the first particle decays, \( T_2 \) the amount of additional time until the second particle decays, etc., so that \( T = T_1 + T_2 + \ldots + T_n \).

Claim: \( T_1 \) is exponential with parameter \( n\lambda \).

Claim: \( T_2 \) is exponential with parameter \( (n - 1)\lambda \).

And so forth. \( E[T] = \sum_{i=1}^{n} E[T_i] = \lambda^{-1} \sum_{j=1}^{n} \frac{1}{j} \) and (by independence) \( \text{Var}[T] = \sum_{i=1}^{n} \text{Var}[T_i] = \lambda^{-2} \sum_{j=1}^{n} \frac{1}{j^2} \).
Outline

Exponential random variables

Minimum of independent exponentials

Memoryless property

Relationship to Poisson random variables
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Relationship to Poisson random variables
Let $T_1, T_2, \ldots$ be independent exponential random variables with parameter $\lambda$. 
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We can view them as waiting times between “events”.

How do you show that the number of events in the first $t$ units of time is Poisson with parameter $\lambda t$?

We actually did this already in the lecture on Poisson point processes. You can break the interval $[0, t]$ into $n$ equal pieces (for very large $n$), let $X_k$ be number of events in $k$th piece, use memoryless property to argue that the $X_k$ are independent.

When $n$ is large enough, it becomes unlikely that any interval has more than one event. Roughly speaking: each interval has one event with probability $\frac{\lambda t}{n}$, zero otherwise.

Take $n \to \infty$ limit. Number of events is Poisson $\lambda t$. 
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18.440 Lecture 20