Outline

Tossing coins

Normal random variables

Special case of central limit theorem
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Normal random variables

Special case of central limit theorem
Suppose we toss a million fair coins. How many heads will we get?
Tossing coins

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- About half a million, yes, but how close to that? Will we be off by 10 or 1000 or 100,000?
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How can we describe the error?
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About half a million, yes, but how close to that? Will we be off by 10 or 1000 or 100,000?

How can we describe the error?

Let’s try this out.
Tossing coins

▶ Toss \( n \) coins. What is probability to see \( k \) heads?

\[
\text{Answer: } 2^n - k \binom{n}{k}.
\]

▶ Let's plot this for a few values of \( n \).

▶ Seems to look like it's converging to a curve.

▶ If we replace fair coin with \( p \) coin, what's probability to see \( k \) heads.

\[
\text{Answer: } p^k (1-p)^{n-k} \binom{n}{k}.
\]

▶ Let's plot this for \( p = \frac{2}{3} \) and some values of \( n \).

What does limit shape seem to be?
Tossing coins

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- Answer: \( 2^{-k} \binom{n}{k} \).
Toss $n$ coins. What is probability to see $k$ heads?

Answer: $2^{-k} \binom{n}{k}$.

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Special case of central limit theorem
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- Tossing coins
- Normal random variables
- Special case of central limit theorem
Say $X$ is a (standard) **normal random variable** if $f_X(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Clearly $f$ is always non-negative for real values of $x$, but how do we show that $\int_{-\infty}^{\infty} f(x) \, dx = 1$? Looks kind of tricky.

Happens to be a nice trick. Write $I = \int_{-\infty}^{\infty} e^{-x^2/2} \, dx$. Then try to compute $I^2$ as a two dimensional integral.

That is, write $I^2 = \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \int_{-\infty}^{\infty} e^{-y^2/2} \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \, dy$.

Then switch to polar coordinates.

$I^2 = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/2} \, r \, d\theta \, dr = 2\pi \int_{0}^{\infty} re^{-r^2/2} \, dr = -2\pi e^{-r^2/2} \bigg|_{0}^{\infty}$, so $I = \sqrt{2\pi}$. 
Standard normal random variable

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\[ I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2/2} dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2/2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2 - y^2/2} dxdy. \]

Switch to polar coordinates:

$$I^2 = \int_{0}^{\infty} \int_{0}^{2\pi} re^{-r^2/2} d\theta dr = 2\pi \int_{0}^{\infty} r e^{-r^2/2} dr.$$ 

Hence

$$I^2 = 2\pi \int_{0}^{\infty} r e^{-r^2/2} dr = \left. -2\pi e^{-r^2/2} \right|_{0}^{\infty} = 2\pi.$$ 

Thus $I = \sqrt{2\pi}$. 

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18.440 Lecture 19
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E[X] = \int_{-\infty}^{\infty} xf(x)\,dx. \text{ Can see by symmetry that this zero.}
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**Or can compute directly:**

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E[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x dx = \frac{1}{\sqrt{2\pi}} \left[ e^{-x^2/2} \right]_{-\infty}^{\infty} = 0.
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Try integration by parts with $u = x$ and $dv = xe^{-x^2/2}\,dx$.

Find that $\text{Var}[X] = \frac{1}{\sqrt{2\pi}} \left( -xe^{-x^2/2} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\,dx ) = 1.$
Again, $X$ is a (standard) **normal random variable** if
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What about $Y = \sigma X + \mu$? Can we “stretch out” and “translate” the normal distribution (as we did last lecture for the uniform distribution)?
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Say $Y$ is normal with parameters $\mu$ and $\sigma^2$ if
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General normal random variables

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$E[Y] = E[X] + \mu = \mu$ and $\text{Var}[Y] = \sigma^2 \text{Var}[X] = \sigma^2$. 

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**General normal random variables**

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*18.440 Lecture 19*
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What is the cumulative distribution function?

Values: $\Phi(-3) \approx 0.0013$, $\Phi(-2) \approx 0.023$ and $\Phi(-1) \approx 0.159$.

Rough rule of thumb: "two thirds of time within one SD of mean, 95 percent of time within 2 SDs of mean."
Again, $X$ is a standard normal random variable if
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What is the cumulative distribution function?

Write this as
\[ F_X(a) = P\{X \leq a\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} \, dx. \]
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DeMoivre-Laplace Limit Theorem

Let $S_n$ be number of heads in $n$ tosses of a $p$ coin.

What's the standard deviation of $S_n$?

Answer: $\sqrt{npq}$ (where $q = 1 - p$).

The special quantity $S_n - np\sqrt{npq}$ describes the number of standard deviations that $S_n$ is above or below its mean.

What's the mean and variance of this special quantity? Is it roughly normal?

DeMoivre-Laplace limit theorem (special case of central limit theorem):

$$\lim_{n \to \infty} P\{a \leq S_n - np\sqrt{npq} \leq b\} \to \Phi(b) - \Phi(a).$$

This is $\Phi(b) - \Phi(a) = P\{a \leq X \leq b\}$ when $X$ is a standard normal random variable.
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Problems

▶ Toss a million fair coins. Approximate the probability that I get more than 501,000 heads.
Problems

▶ Toss a million fair coins. Approximate the probability that I get more than 501,000 heads.

▶ Answer: well, \( \sqrt{npq} = \sqrt{10^6 \times .5 \times .5} = 500 \). So we’re asking for probability to be over two SDs above mean. This is approximately \( 1 - \Phi(2) = \Phi(-2) \approx .159 \).
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Roll 60000 dice. Expect to see 10000 sixes. What’s the probability to see more than 9800?
Problems

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- Answer: well, $\sqrt{npq} = \sqrt{10^6 \times .5 \times .5} = 500$. So we’re asking for probability to be over two SDs above mean. This is approximately $1 - \Phi(2) = \Phi(-2) \approx .159$.
- Roll 60000 dice. Expect to see 10000 sixes. What’s the probability to see more than 9800?
  - Here $\sqrt{npq} = \sqrt{60000 \times \frac{1}{6} \times \frac{5}{6}} \approx 91.28$. 
Problems

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▶ Roll 60000 dice. Expect to see 10000 sixes. What’s the probability to see more than 9800?

▶ Here $\sqrt{npq} = \sqrt{60000 \times \frac{1}{6} \times \frac{5}{6}} \approx 91.28$.

▶ And $200/91.28 \approx 2.19$. Answer is about $1 - \Phi(-2.19)$. 

18.440 Lecture 19