# 18.440: Lecture 12

# Poisson random variables

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# Outline

Poisson random variable definition

Poisson random variable properties

Poisson random variable problems

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# Poisson random variables: motivating questions

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- **Key idea for all these examples:** Divide time into large number of small increments. Assume that during each increment, there is some small probability of thing happening (independently of other increments).

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- Can also change sign: $e^{-\lambda} = \lim_{n\to\infty}(1 - \lambda/n)^n$.

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- Use Taylor expansion $e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$.

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- Setting $j = k - 1$, this is $\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}e^{-\lambda} = \lambda$.

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- Then $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda$.

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- A casino deals one million five-card poker hands per year. Approximate the probability that there are exactly 2 royal flush hands during a given year.