

18.440: Lecture 11

Binomial random variables and repeated trials

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Bernoulli random variables

Properties: expectation and variance

More problems

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- ▶ Can use binomial theorem to show probabilities sum to one:
- ▶ $1 = 1^n = (p + q)^n = \sum_{k=0}^n \binom{n}{k}p^kq^{n-k}$.
- ▶ Number of heads is **binomial random variable with parameters (n, p)** .

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- ▶ Probability mass function for X can be computed using the 6th row of Pascal's triangle.
- ▶ If coin is biased (comes up heads with probability $p \neq 1/2$), we can still use the 6th row of Pascal's triangle, but the probability that $X = i$ gets multiplied by $p^i(1 - p)^{n-i}$.

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- ▶ Let $n = 100$. Compute the probability that nobody was born on a Tuesday.
- ▶ What is the probability that exactly 15 people were born on a Tuesday?

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- ▶ Perhaps the prior row $(1, 4, 6, 4, 1)$?

Useful Pascal's triangle identity

- ▶ Recall that $\binom{n}{i} = \frac{n \times (n-1) \times \dots \times (n-i+1)}{i \times (i-1) \times \dots \times (1)}$. This implies a simple but important identity: $i \binom{n}{i} = n \binom{n-1}{i-1}$.

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- ▶ Using this identity (and $q = 1 - p$), we can write

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- ▶ Substitute $j = i - 1$ to get

$$E[X] = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j} = np(p+q)^{n-1} = np.$$

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- ▶ Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^n E[X_j] = \sum_{j=1}^n p = np.$$

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- ▶ Thus $E[X^k] = npE[(Y+1)^{k-1}]$ where Y is binomial with parameters $(n-1, p)$.

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- ▶ Commit to memory: variance of binomial (n, p) random variable is npq .
- ▶ This is n times the variance you'd get with a single coin. Coincidence?

Compute variance with decomposition trick

- ▶ $X = \sum_{j=1}^n X_j$, so
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- ▶ You invite 50 friends to a party. Each one, independently, has a $1/3$ chance of showing up. What is the probability that more than 25 people will show up?