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Decomposition trick
Recall: a random variable $X$ is a function from the state space to the real numbers.
Recall definitions for expectation

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- Can interpret $X$ as a quantity whose value depends on the outcome of an experiment.
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- Can interpret \( X \) as a quantity whose value depends on the outcome of an experiment.
- Say \( X \) is a \textbf{discrete} random variable if (with probability one) it takes one of a countable set of values.

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E[X] = \sum_{x \in \text{domain of } P} x \cdot P(X=x)
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Also,
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E[g(X)] = \sum_{x \in \text{domain of } P} g(x) \cdot P(X=x)
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$$E[X] = \sum_{x: p(x) > 0} xp(x).$$
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E[g(X)] = \sum_{x: p(x) > 0} g(x) p(x),
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we find that
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Original formula gives intuitive idea of what variance is (expected square of difference from mean). But we will often use this alternate formula when we have to actually compute the variance.
Let $X$ be a random variable with mean $\mu$.

We introduced above the formula $\text{Var}(X) = E[(X - \mu)^2]$.

This can be written $\text{Var}(X) = E[X^2] - 2\mu E[X] + \mu^2$.

By additivity of expectation, this is the same as $E[X^2] - 2\mu^2 = E[X^2] - \mu^2$.

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Let $Y$ be number of heads in two fair coin tosses. What is $\text{Var}[Y]$?
Variance examples

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  Recall $P\{Y = 0\} = 1/4$ and $P\{Y = 1\} = 1/2$ and $P\{Y = 2\} = 1/4$. 
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  Then $\text{Var}[Y] = E[Y^2] - E[Y]^2 = \frac{1}{4}0^2 + \frac{1}{2}1^2 + \frac{1}{4}2^2 - 1^2 = \frac{1}{2}$.  

More variance examples

- You buy a lottery ticket that gives you a one in a million chance to win a million dollars.
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- At a particular party, there are four five-foot-tall people, five six-foot-tall people, and one seven-foot-tall person. You pick one of these people uniformly at random. What is the expected height of the person you pick?
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Identity

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Identity

- If $Y = X + b$, where $b$ is constant, then does it follow that $\text{Var}[Y] = \text{Var}[X]$?
- Yes.
- We showed earlier that $E[aX] = aE[X]$. We claim that $\text{Var}[aX] = a^2\text{Var}[X]$. 

18.440 Lecture 10
If $Y = X + b$, where $b$ is constant, then does it follow that $\text{Var}[Y] = \text{Var}[X]$?

Yes.

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18.440 Lecture 10
Write $SD[X] = \sqrt{Var[X]}$. 
Standard deviation

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If we switch from feet to inches in our "height of randomly chosen person" example, then $X$, $E[X]$, and $SD[X]$ each get multiplied by 12, but $Var[X]$ gets multiplied by 144.
Standard deviation

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Number of aces

Choose five cards from a standard deck of 52 cards. Let $A$ be the number of aces you see.
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Answer: $\binom{52}{5}$.
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Answer: $\binom{52}{5}$.

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Answer: $\binom{4}{k} \binom{48}{5-k}$.
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So $P\{A = k\} = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}}$. 
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Number of aces

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  - Answer: $\binom{4}{k}\binom{48}{5-k}$.
- So $P\{A = k\} = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}}$.
- So $E[A] = \sum_{k=0}^{4} kP\{A = k\}$,
- and $\text{Var}[A] = \sum_{k=0}^{4} k^2P\{A = k\} - E[A]^2$. 

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Then $A = \sum_{i=1}^{5} A_i$. And $E[A] = \sum_{i=1}^{5} E[A_i] = \frac{5}{13}$.

$A^2 = (A_1 + A_2 + \ldots + A_5)^2$ can be expanded into 25 terms:

$A^2 = \sum_{i=1}^{5} \sum_{j=1}^{5} A_i A_j$.

So $E[A^2] = \sum_{i=1}^{5} \sum_{j=1}^{5} E[A_i A_j]$.

Five terms of form $E[A_i A_j]$ with $i = j$ five with $i \neq j$. First five contribute $\frac{1}{13}$ each. How about other twenty?

$E[A_i A_j] = \left(\frac{1}{13}\right) \left(\frac{3}{51}\right) = \left(\frac{1}{13}\right) \left(\frac{1}{17}\right)$. So $E[A^2] = \frac{5}{13} + \frac{20}{13} \times \frac{17}{17} = \frac{105}{13} \times \frac{17}{17}$.

$\text{Var}[A] = E[A^2] - E[A]^2 = \frac{105}{13} \times \frac{17}{17} - \frac{25}{13} \times \frac{13}{13}$.
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$E[A_i A_j] = (1/13)(3/51) = (1/13)(1/17)$. So $E[A^2] = \frac{5}{13} + \frac{20}{13 \times 17} = \frac{105}{13 \times 17}$. 

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$\text{Var}[A] = E[A^2] - E[A]^2 = \frac{105}{13 \times 17} - \frac{25}{13 \times 13}$. 
In the \( n \)-hat shuffle problem, let \( X \) be the number of people who get their own hat. What is \( \text{Var}[X] \)?

We showed earlier that \( \mathbb{E}[X] = 1 \). So \( \text{Var}[X] = \mathbb{E}[X^2] - 1 \). But how do we compute \( \mathbb{E}[X^2] \)?

Decomposition trick: write variable as sum of simple variables.

Let \( X_i \) be one if \( i \)th person gets own hat and zero otherwise. Then \( X = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i \).

We want to compute \( \mathbb{E}[(X_1 + X_2 + \ldots + X_n)^2] \).

Expand this out and using linearity of expectation:

\[
\mathbb{E}\left[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] \sum_{j=1}^{n} \mathbb{E}[X_j] = n \cdot 1 + n(n-1) \frac{1}{n(n-1)} = 2.
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So \( \text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 2 - 1 = 1. \)
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$$E[\sum_{i=1}^n X_i \sum_{j=1}^n X_j] = \sum_{i=1}^n \sum_{j=1}^n E[X_i X_j] = n \cdot \frac{1}{n} + n(n-1) \frac{1}{n(n-1)} = 2.$$
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Decomposition trick: write variable as sum of simple variables.

Let $X_i$ be one if $i$th person gets own hat and zero otherwise. Then $X = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^n X_i$.

We want to compute $E[(X_1 + X_2 + \ldots + X_n)^2]$.

Expand this out and using linearity of expectation:

$$E[\sum_{i=1}^n X_i \sum_{j=1}^n X_j] = \sum_{i=1}^n \sum_{j=1}^n E[X_i X_j] = n \cdot \frac{1}{n} + n(n-1) \frac{1}{n(n-1)} = 2.$$

So $\text{Var}[X] = E[X^2] - (E[X])^2 = 2 - 1 = 1$. 