18.440: Lecture 10

Variance and standard deviation

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Outline

Defining variance

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Decomposition trick

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Recall definitions for expectation

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- Also,

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x).$$

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- Variance is one way to measure the amount a random variable "varies" from its mean over successive trials.

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- This gives us our very important alternate formula: $\mathrm{Var}[X] = E[X^2] - (E[X])^2$.
- Original formula gives intuitive idea of what variance is (expected square of difference from mean). But we will often use this alternate formula when we have to actually compute the variance.

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- Yes.
- We showed earlier that $E[aX] = aE[X]$. We claim that $\mathrm{Var}[aX] = a^2\mathrm{Var}[X]$.
- Proof: $\mathrm{Var}[aX] = E[a^2X^2] - E[aX]^2 = a^2E[X^2] - a^2E[X]^2 = a^2\mathrm{Var}[X]$.

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- If we switch from feet to inches in our "height of randomly chosen person" example, then X, $E[X]$, and $\mathrm{SD}[X]$ each get multiplied by 12, but $\mathrm{Var}[X]$ gets multiplied by 144.

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- So $E[A] = \sum_{k=0}^4 kP\{A=k\}$,
- and $\mathrm{Var}[A] = \sum_{k=0}^4 k^2 P\{A=k\} - E[A]^2$.

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- $E[A_i A_j] = (1/13)(3/51) = (1/13)(1/17)$. So $E[A^2] = \frac{5}{13} + \frac{20}{13\times 17} = \frac{105}{13\times 17}$.

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- $\mathrm{Var}[A] = E[A^2] - E[A]^2 = \frac{105}{13\times 17} - \frac{25}{13\times 13}$.

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- So $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = 2 - 1 = 1$.