Permutations and combinations, Pascal’s triangle, learning to count

Scott Sheffield

MIT
Remark, just for fun

Permutations

Counting tricks

Binomial coefficients

Problems
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Problems
Suppose that betting markets place the probability that your favorite presidential candidates will be elected at 58 percent. Price of a contact that pays 100 dollars if your candidate wins is 58 dollars.
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Let $X(t)$ denote the price at time $t$.

Suppose $X(t)$ is known to vary continuously in time. What is the probability it will reach 59 before reaching 57?
Politics

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- “Efficient market hypothesis” suggests about .5.
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- Suppose $X(t)$ is known to vary continuously in time. What is the probability it will reach 59 before reaching 57?
- “Efficient market hypothesis” suggests about .5.
- Reasonable model: use sequence of fair coin tosses to decide the order in which $X(t)$ passes through different integers.
Which of these statements is “probably” true?

1. $X(t)$ will go below 50 at some future point.
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- 2. $X(t)$ will get all the way below 20 at some point.
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- 1. $X(t)$ will go below 50 at some future point.
- 2. $X(t)$ will get all the way below 20 at some point.
- 3. $X(t)$ will reach both 70 and 30, at different future times.

Answers: 1, 2, 4.

Full explanations coming at the end of the course.

Point for now is that probability is everywhere: politics, military, finance and economics, all kinds of science and engineering, philosophy, religion, making cool new cell phone features work, social networking, dating websites, etc.

All of the math in this course has a lot of applications.
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4. \( X(t) \) will reach both 65 and 35 at different future times.
5. \( X(t) \) will hit 65, then 50, then 60, then 55.

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Outline

Remark, just for fun

Permutations

Counting tricks

Binomial coefficients

Problems
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Problems
Permutations

- How many ways to order 52 cards?

\[ 52 \cdot 51 \cdot 50 \cdot \ldots \cdot 1 = 52! = 80658175170943878571660636856403766975289505440883277824 \times 10^{12} \]

- \( n \) hats, \( n \) people, how many ways to assign each person a hat?

\[ n! \]

- \( n \) hats, \( k < n \) people, how many ways to assign each person a hat?

\[ n \cdot (n-1) \cdot (n-2) \ldots (n-k+1) = \frac{n!}{(n-k)!} \]

- A permutation is a map from \( \{1, 2, \ldots, n\} \) to \( \{1, 2, \ldots, n\} \).

There are \( n! \) permutations of \( n \) elements.
How many ways to order 52 cards?

Answer: $52 \cdot 51 \cdot 50 \cdot \ldots \cdot 1 = 52! = 80658175170943878571660636856403766975289505440883277824 \times 10^{12}$
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A permutation is a map from \{1, 2, \ldots, n\} to \{1, 2, \ldots, n\}. There are $n!$ permutations of $n$ elements.
A permutation is a function from \( \{1, 2, \ldots, n\} \) to \( \{1, 2, \ldots, n\} \) whose range is the whole set \( \{1, 2, \ldots, n\} \). If \( \sigma \) is a permutation then for each \( j \) between 1 and \( n \), the value \( \sigma(j) \) is the number that \( j \) gets mapped to.
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For example, if \( n = 3 \), then \( \sigma \) could be a function such that \( \sigma(1) = 3, \sigma(2) = 2, \) and \( \sigma(3) = 1 \).
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If you have \( n \) cards with labels 1 through \( n \) and you shuffle them, then you can let \( \sigma(j) \) denote the label of the card in the \( j \)th position. Thus orderings of \( n \) cards are in one-to-one correspondence with permutations of \( n \) elements.
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One way to represent \( \sigma \) is to list the values \( \sigma(1), \sigma(2), \ldots, \sigma(n) \) in order. The \( \sigma \) above is represented as \( \{3, 2, 1\} \).
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If \( \sigma \) and \( \rho \) are both permutations, write \( \sigma \circ \rho \) for their composition. That is, \( \sigma \circ \rho(j) = \sigma(\rho(j)) \).
Another way to write a permutation is to describe its cycles:

For example, taking \( n = 7 \), we write \((2, 3, 5), (1, 7), (4, 6)\) for the permutation \( \sigma \) such that \( \sigma(2) = 3, \sigma(3) = 5, \sigma(5) = 2 \) and \( \sigma(1) = 7, \sigma(7) = 1, \sigma(4) = 6 \).

If you pick some \( j \) and repeatedly apply \( \sigma \) to it, it will "cycle through" the numbers in its cycle.

Generally, a function is called an involution if \( f(f(x)) = x \) for all \( x \).

A permutation is an involution if all cycles have length one or two.

A permutation is "fixed point free" if there are no cycles of length one.
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Outline

Remark, just for fun

Permutations

Counting tricks

Binomial coefficients

Problems
Outline

Remark, just for fun

Permutations

**Counting tricks**

Binomial coefficients

Problems
Fundamental counting trick

- $n$ ways to assign hat for the first person. No matter what choice I make, there will remain $n - 1$ ways to assign hat to the second person. No matter what choice I make there, there will remain $n - 2$ ways to assign a hat to the third person, etc.
Fundamental counting trick

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- This is a useful trick: break counting problem into a sequence of stages so that one always has the same number of choices to make at each stage. Then the total count becomes a product of number of choices available at each stage.
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- This is a useful trick: break counting problem into a sequence of stages so that one always has the same number of choices to make at each stage. Then the total count becomes a product of number of choices available at each stage.

- Easy to make mistakes. For example, maybe in your problem, the number of choices at one stage actually does depend on choices made during earlier stages.
Another trick: overcount by a fixed factor

If you have 5 indistinguishable black cards, 2 indistinguishable red cards, and three indistinguishable green cards, how many distinct shuffle patterns of the ten cards are there?

Answer: if the cards were distinguishable, we'd have 10!. But we're overcounting by a factor of 5!2!3!, so the answer is $\frac{10!}{5!2!3!}$. 
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- Remark, just for fun
- Permutations
- Counting tricks
- Binomial coefficients
- Problems
Outline

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Permutations

Counting tricks

Binomial coefficients

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\( \binom{n}{k} \) notation

- How many ways to choose an ordered sequence of \( k \) elements from a list of \( n \) elements, with repeats allowed?

Answer: \( n^k \)

- How many ways to choose an ordered sequence of \( k \) elements from a list of \( n \) elements, with repeats forbidden?

Answer: \( \frac{n!}{(n-k)!} \)

- How many ways to choose (unordered) \( k \) elements from a list of \( n \) without repeats?

Answer: \( \binom{n}{k} := \frac{n!}{k!(n-k)!} \)

- What is the coefficient in front of \( x^k \) in the expansion of \( (x+1)^n \)?

Answer: \( \binom{n}{k} \).
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$\binom{n}{k}$ notation
Pascal’s triangle

- Arnold principle.

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

What is the coefficient in front of \(x^k\) in the expansion of \((x+1)^n\)?

Answer: \(\binom{n}{k}\).

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(x+1)^n = \binom{n}{0} \cdot 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n.
\]

Question: what is \(\sum_{k=0}^{n} \binom{n}{k}\)?

Answer: \((1 + 1)^n = 2^n\).
Pascal’s triangle

- Arnold principle.
- A simple recursion: \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \).

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\(18.440 \text{ Lecture 1} \)
Arnold principle.

A simple recursion: \((\binom{n}{k}) = (\binom{n-1}{k-1}) + (\binom{n-1}{k})\).

What is the coefficient in front of \(x^k\) in the expansion of \((x + 1)^n\)?

Answer: \((\binom{n}{k})\).

\((x + 1)^n = (\binom{n}{0}) \cdot 1 + (\binom{n}{1})x^1 + (\binom{n}{2})x^2 + \ldots + (\binom{n}{n-1})x^{n-1} + (\binom{n}{n})x^n\).

Question: what is \(\sum_{k=0}^{n} \binom{n}{k}\)?

Answer: \((1 + 1)^n = 2^n\).
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Problems
More problems

- How many full house hands in poker?
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- \(13 \binom{4}{3} \cdot 12 \binom{4}{2}\)
More problems

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More problems

- How many hands that have four cards of the same suit, one card of another suit?
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More problems

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  $$4 \binom{13}{4} \cdot 3 \binom{13}{1}$$
- How many 10 digit numbers with no consecutive digits that agree?
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  - If initial digit can be zero, have \( 10 \cdot 9^9 \) ten-digit sequences. If initial digit required to be non-zero, have \( 9^{10} \).
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- How many 10 digit numbers (allowing initial digit to be zero) in which only 5 of the 10 possible digits are represented?

This is one is tricky, can be solved with inclusion-exclusion (to come later in the course).
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- \(366^{23}\) if repeats allowed. \(366!/343!\) if repeats not allowed.