18.440 Midterm 2, Spring 2014: 50 minutes, 100 points

1. (20 points) Consider a sequence of independent tosses of a coin that is biased so that it comes up heads with probability 3/4 and tails with probability 1/4. Let X_i be 1 if the ith toss comes up heads and 0 otherwise.

(a) Compute $E[X_1]$ and $\mathrm{Var}[X_1]$. **ANSWER:** $E[X_1] = 3/4$ and $E[X_1^2] = 3/4$ so

$$\mathrm{Var}[X_1] = E[X^2] - E[X]^2 = (3/4) - (3/4)^2 = (3/4)(1/4) = 3/16.$$

(b) Compute $\mathrm{Var}[X_1 + 2X_2 + 3X_3 + 4X_4]$. **ANSWER:** Using previous problem, additivity of variance for independent random variables, and general fact that $\mathrm{Var}[aY] = a^2\mathrm{Var}[Y]$, we find that

$$\mathrm{Var}[X_1 + 2X_2 + 3X_3 + 4X_4] = (3/16)(1 + 4 + 9 + 16) = 90/16 = 45/8.$$

(c) Let Y be the number of heads in the first 4800 tosses of the biased coin, i.e.,

$$Y = \sum_{i=1}^{4800} X_i.$$

Use a normal random variable to approximate the probability that $Y \geq 3690$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer. **ANSWER:** Y has expectation $4800E[X_1] = 3600$. It has variance $4800\mathrm{Var}[X_1] = 900$ and standard deviation 30. We are are looking for the probability that Y is more than three standard deviations above its mean. This is approximately the probability that standard normal random variable is three standard deviations above its mean, which is $1 - \Phi(3)$.

2. (10 points) Suppose that a fair six-sided die is rolled just once. Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the number that comes up. Let Y be 1 if the number on the die is in $\{1, 2, 3\}$ and 0 otherwise.

(a) What is the conditional expectation of X given that $Y = 0$? **ANSWER:** Given that Y is zero, X is conditionally uniform on $\{4, 5, 6\}$, so the conditional expectation is 5.

(b) What is the conditional variance of Y given that $X = 2$? **ANSWER:** Given that X is 2, the conditional probability that $Y = 1$ is one, so the conditional variance is 0.

3. (20 points) Let X be a uniform random variable on the set $\{-2, -1, 0, 1, 2\}$. That is, X takes each of these values with probability $1/5$. Let Y be an independent random variable with the same law as X, and write $Z = X + Y$.

(a) What is the moment generating function $M_X(t)$? **ANSWER:** $M_X(t) = E[e^{tX}] = \frac{1}{5}(e^{-2t} + e^{-t} + e^0 + e^t + e^{2t})$.

(b) What is the moment generating function $M_Z(t)$? **ANSWER:**

$$M_Z(t) = M_X(t)M_Y(t) = \big[\frac{1}{5}(e^{-2t} + e^{-t} + e^0 + e^t + e^{2t})\big]^2.$$

4. (20 points) Two soccer teams, the Lions and the Tigers, begin an infinite soccer games starting at time zero. Suppose that the times at which the Lions score a goal form a Poisson point process with rate $\lambda_L = 2$/hour. Suppose that the times at which the Tigers score a goal form a Poisson point process with rate $\lambda_T = 3$/hour.

(a) Write down the probability density function for the amount of time until the first goal by the Lions. **ANSWER:** This is an exponential random variable with parameter λ_L. So the density function on $[0, \infty)$ is $f(x) = \lambda_L e^{-\Lambda_L x} = 2e^{-2x}$.

(c) Write down the probability density function for the amount of time until the first goal by *either* team is scored. **ANSWER:** Recall that the minimum of two exponential random variables with parameters λ_L and λ_T is an exponential random variable with parameter $\lambda_L + \Lambda_T = 5$. So the density function on $[0, \infty)$ is $f(x) = 5e^{-5x}$

(c) Compute the probability that the Tigers score no goals at all during the first two hours. **ANSWER:** The probability that an exponential random of parameter λ is at least a is given by $e^{-\lambda a}$. Plugging in $\lambda = 3$ and $a = 2$ we get e^{-6}.

(d) Compute the probability that the Lions score exactly three goals during the first hour. **ANSWER:** The number of goals scored by the Lions during the first hour is a Poisson random variable with parameter $\lambda = \lambda_L = 2$. The probability that this is equal to a given k is given by $e^{-\lambda}\lambda^k/k!$. Plugging in $k = 3$ and $\lambda = 2$ we get

$$e^{-2}2^3/3! = \frac{4}{3e^2}.$$

5. (20 points) Let X and Y be independent uniform random variables on $[0,1]$. Write $Z = X + Y$. Write $W = \max\{X, Y\}$.

(a) Compute and draw a graph of the probability density function f_Z. **ANSWER:** This is given by

$$f_Z(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 2 - x & 1 < x \le 2 \\ 0 & x \ge 2 \end{cases}$$

(b) Compute and draw a graph of the cumulative distribution function F_W. **ANSWER:** $F_W(a) = \begin{cases} 0 & a < 0 \\ a^2 & 0 \le a \le 1 \\ 1 & a > 1 \end{cases}$

(c) Compute the variances $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Var}(Z)$. **ANSWER:** $\mathrm{Var}(X) = E[X^2] - E[X]^2 = \int_0^1 x^2 dx - (1/2)^2 = 1/3 - 1/4 = 1/12$. Then $\mathrm{Var}(Y) = \mathrm{Var}(X)$ and $\mathrm{Var}(Z) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 2/12$.

(d) Compute the covariance $\mathrm{Cov}(Y, Z)$ and the correlation coefficient $\rho(Y, Z)$. **ANSWER:** Using the linearity of covariance in its second argument, we find $\mathrm{Cov}(Y, Z) = \mathrm{Cov}(Y, X) + \mathrm{Cov}(Y, Y)$. The first term is zero (since X and Y are independent) so this becomes $\mathrm{Var}(Y) = 1/12$. The correlation coefficient is

$$\frac{\mathrm{Cov}(Y, Z)}{\sqrt{\mathrm{Var}(Y)\mathrm{Var}(Z)}} = \frac{(1/12)}{\sqrt{(1/12)(2/12)}} = 1/\sqrt{2}.$$

6. (10 points) Let X and Y be independent exponential random variables, each with with parameter $\lambda = 5$.

(a) Let f be the joint probability density function for the pair (X, Y). Write an explicit formula for f. **ANSWER:** Since X and Y are independent, $f(x, y) = f_X(x) f_Y(y) = 5e^{-5x} \cdot 5e^{-5y} = 25e^{-5(x+y)}$.

(b) Compute $E[X^2 Y]$. **ANSWER:** First, note that X^2 and Y are independent, so this is $E[X^2]E[Y]$. Direct integration gives $E[Y] = 1/\lambda$ and $E[X^2] = 2/\lambda^2$, so the answer is $2/\lambda^3 = 2/125$.