

18.440 Midterm 1, Fall 2012: 50 minutes, 100 points

1. Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.
2. No calculators, books, or notes may be used.
3. Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).

1. (20 points) Toss 6 independent fair coins, and let  $X$  be the number that come up heads. Compute the following:

(a)  $E[X]$

(b)  $\text{Var}[X]$

(c) The conditional probability  $P[X = 6 | X \geq 5]$ .

2. (20 points) Four people have two hats each. They take off these hats and shuffle them up randomly and redistribute them among the four people (so that each again has two hats). Assume that all the ways of dividing the 8 hats among the 4 people (with each person getting two hats) are equally likely.

(a) How many possible outcomes are there (i.e., how many ways are there to divide 8 distinct hats among four distinct people, with each person getting two hats)?

(b) Label the people 1, 2, 3, and 4. Let  $E_i$  be the event that the  $i$ th person gets *both* of his or her own hats. Compute  $P[E_1]$ .

(c) Compute  $P[E_1E_2]$ . Then compute  $P[E_1E_2E_3]$  (which is the same as  $P[E_1E_2E_3E_4]$ ).

(d) Write  $A = P[E_1]$ ,  $B = P[E_1E_2]$ ,  $C = P[E_1E_2E_3] = P[E_1E_2E_3E_4]$ . Use inclusion-exclusion to express  $P[E_1 \cup E_2 \cup E_3 \cup E_4]$  in terms of  $A$ ,  $B$ , and  $C$ .

3. (20 points) Suppose that during each minute of a 90-minute soccer game there is a probability of  $2/90$  that one goal will be scored and a probability of  $88/90$  that no goal will be scored (independently of all other minutes). Let  $N$  be the total number of goals scored during the game.

(a) Compute  $E[N]$  and  $\text{Var}[N]$ . (Give exact answers, not approximate ones.)

(b) Compute the probability that there is exactly one goal. Give an *exact* answer.

(c) Let  $E$  be the event that there are exactly 0 goals in the first half and exactly 2 goals in the second half. Use a Poisson random variable calculation to *approximate* the probability of  $E$ .

4. (10 points) Roll 6 independent fair six-sided dice (so that each of the  $6^6$  outcomes is equally likely). Compute the probability that exactly three of the dice land on even numbers.

5. (20 points) Roll 2 independent fair six-sided dice. Let  $X$  be the value on the first die,  $Y$  the value on the second die. Write  $Z = Y + X$ . Compute the following:

(a)  $E[Z]$ .

(b)  $E[Y^2]$ .

(c)  $E[YZ]$ .

6. (10 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability  $1/3$ . Let  $X$  be such that the third heads occurs on the  $X$ th toss.

(a) Compute  $P[X = 9]$ .

(b) Compute  $E[X]$ .