# 18.600: Lecture 14 Lectures 1-13 Review

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#### Counting tricks and basic principles of probability

Discrete random variables

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Discrete random variables

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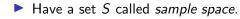
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- Answer: <sup>n+k-1</sup> n. Represent partition by k - 1 bars and n stars, e.g., as \*\* | \*\*|| \*\*\*\*|\*.



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- Countable additivity:  $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$  if  $E_i \cap E_j = \emptyset$  for each pair *i* and *j*.

▶ 
$$P(A^c) = 1 - P(A)$$

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$$P(AB) \leq P(A)$$

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More generally,

$$P(\bigcup_{i=1}^{n} E_{i}) = \sum_{i=1}^{n} P(E_{i}) - \sum_{i_{1} < i_{2}} P(E_{i_{1}}E_{i_{2}}) + \dots + (-1)^{(r+1)} \sum_{i_{1} < i_{2} < \dots < i_{r}} P(E_{i_{1}}E_{i_{2}} \dots E_{i_{r}})$$
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► The notation ∑<sub>i1<i2<...<ir</sub> means a sum over all of the <sup>n</sup><sub>r</sub> subsets of size r of the set {1, 2, ..., n}.

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   Answer: 1/1.

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- Answer: <sup>1</sup>/<sub>r!</sub>.
   P(\u2297\_{i=1}^n E\_i) = 1 <sup>1</sup>/<sub>2!</sub> + <sup>1</sup>/<sub>3!</sub> <sup>1</sup>/<sub>4!</sub> + \u2295 ± <sup>1</sup>/<sub>n!</sub>

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• Answer:  $\frac{1}{r!}$ .

- $P(\cup_{i=1}^{n} E_i) = 1 \frac{1}{2!} + \frac{1}{3!} \frac{1}{4!} + \dots \pm \frac{1}{n!}$
- ►  $1 P(\bigcup_{i=1}^{n} E_i) = 1 1 + \frac{1}{2!} \frac{1}{3!} + \frac{1}{4!} \ldots \pm \frac{1}{n!} \approx 1/e \approx .36788$

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- Nice fact:  $P(E_1E_2E_3...E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)...P(E_n|E_1...E_{n-1})$

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- Useful when we think about multi-step experiments.
- For example, let E<sub>i</sub> be event ith person gets own hat in the n-hat shuffle problem.

## $P(E) = P(EF) + P(EF^{c})$ = $P(E|F)P(F) + P(E|F^{c})P(F^{c})$

## Dividing probability into two cases

## $P(E) = P(EF) + P(EF^{c})$ = $P(E|F)P(F) + P(E|F^{c})P(F^{c})$

In words: want to know the probability of *E*. There are two scenarios *F* and *F<sup>c</sup>*. If I know the probabilities of the two scenarios and the probability of *E* conditioned on each scenario, I can work out the probability of *E*.

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▶ Ratio  $\frac{P(B|A)}{P(B)}$  determines "how compelling new evidence is".

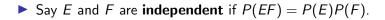
▶ We can check the probability axioms:  $0 \le P(E|F) \le 1$ , P(S|F) = 1, and  $P(\cup E_i) = \sum P(E_i|F)$ , if *i* ranges over a countable set and the  $E_i$  are disjoint.

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- To get former from latter, we set probabilities of elements outside of F to zero and multiply probabilities of events inside of F by 1/P(F).
- ▶ P(·) is the prior probability measure and P(·|F) is the posterior measure (revised after discovering that F occurs).



- Say *E* and *F* are **independent** if P(EF) = P(E)P(F).
- Equivalent statement: P(E|F) = P(E). Also equivalent: P(F|E) = P(F).

Say 
$$E_1 \dots E_n$$
 are independent if for each  $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots n\}$  we have  $P(E_{i_1}E_{i_2}\dots E_{i_k}) = P(E_{i_1})P(E_{i_2})\dots P(E_{i_k}).$ 

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- Does pairwise independence imply independence?
- No. Consider these three events: first coin heads, second coin heads, odd number heads. Pairwise independent, not independent.

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- For each a in this countable set, write p(a) := P{X = a}.
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- ▶ Write  $F(a) = P\{X \le a\} = \sum_{x \le a} p(x)$ . Call *F* the cumulative distribution function.

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- Example: in *n*-hat shuffle problem, let *E<sub>i</sub>* be the event *i*th person gets own hat.
- Then  $\sum_{i=1}^{n} 1_{E_i}$  is total number of people who get own hats.

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Represents weighted average of possible values X can take, each value being weighted by its probability. If the state space S is countable, we can give SUM OVER STATE SPACE definition of expectation:

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Agrees with the SUM OVER POSSIBLE X VALUES definition:

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- ▶ How can we compute *E*[*g*(*X*)]?

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Answer:

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x).$$

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- > This is called the **linearity of expectation**.
- Can extend to more variables  $E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n].$

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- ► Taking  $g(x) = (x \mu)^2$ , and recalling that  $E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$ , we find that

$$\operatorname{Var}[X] = \sum_{x: p(x) > 0} (x - \mu)^2 p(x).$$

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- Very important alternate formula:  $Var[X] = E[X^2] (E[X])^2$ .

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• Proof:  $\operatorname{Var}[aX] = E[a^2X^2] - E[aX]^2 = a^2 E[X^2] - a^2 E[X]^2 = a^2 \operatorname{Var}[X].$ 

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- If we switch from feet to inches in our "height of randomly chosen person" example, then X, E[X], and SD[X] each get multiplied by 12, but Var[X] gets multiplied by 144.

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- Note that  $E[X_j] = p \cdot 1 + (1-p) \cdot 0 = p$  for each j.
- Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} p = np.$$

•  $X = \sum_{j=1}^{n} X_j$ , so  $E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{i=1}^{n} E[X_i X_j]$ 

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• Can show generally that if  $X_1, \ldots, X_n$  independent then  $\operatorname{Var}[\sum_{j=1}^n X_j] = \sum_{j=1}^n \operatorname{Var}[X_j]$ 

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- The numbers of events occurring in disjoint intervals are independent random variables.
- Probability to see zero events in first t time units is  $e^{-\lambda t}$ .
- Let *T<sub>k</sub>* be time elapsed, since the previous event, until the *k*th event occurs. Then the *T<sub>k</sub>* are independent random variables, each of which is exponential with parameter *λ*.

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- What is the probability of a bridge hand with 3 of one suit, 3 of one suit, 2 of one suit, 5 of another suit?

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- $P(\text{disease}|\text{positive}) = \frac{.9p}{.9p+.1(1-p)}$ . If p is tiny, this is about 9p.