Fall 2023 18.600 Final Exam Solutions

1.(10 points) Carl is preparing for an important biology exam. While attempting to focus, Carl has six states: (i) diligently studying, (ii) browsing social media, (iii) following the news, (iv) reading email, (v) answering text messages, and (vi) absentmindedly checking the refrigerator. Every 60 seconds, Carl randomly updates his state as follows: if he is in state (i) — studying — he stays there with probability 5/6 and switches to each of the other 5 states with probability 1/30. If he is in one of states (ii) to (vi), he stays in that state with probability 1/2 and switches to each of the other 5 with probability 1/10.

(a) Write the transition matrix M corresponding to Carl's update procedure. **ANSWER:**

$$\begin{pmatrix} 5/6 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 \\ 1/10 & 1/2 & 1/10 & 1/10 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/2 & 1/10 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/10 & 1/2 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/10 & 1/10 & 1/2 & 1/10 \\ 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/2 \end{pmatrix}$$

(b) If Carl is diligently studying now, what is the expected number of updates until he first switches to a different state (and is no longer studying)? **ANSWER:** Each step Carl has 1/6 chance to stop studying. Number of updates needed is geometric with parameter 1/6. Expectation is 6.

(c) If Carl is browsing social media now, what is the expected number of updates until Carl is studying again? **ANSWER:** Each time Carl is not studying, there is 1/10 chance to start studying. Number of updates needed is geometric with parameter 1/10. Expectation is 10.

(d) Over the long haul, what fraction of the time does Carl spend in each of the six states? **ANSWER:** Using (b) and (c) one can see that over the long haul, Carl is studying a $6/(6+10) = 3/8$ fraction of the time, and by symmetry spends a 1/8 fraction of time in each of the other states. Alternative: use that (ii) to (vi) have same probability by symmetry and solve:

$$\begin{pmatrix} x & y & y & y & y & y \end{pmatrix} \begin{pmatrix} 5/6 & 1/30 & 1/30 & 1/30 & 1/30 & 1/30 \\ 1/10 & 1/2 & 1/10 & 1/10 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/2 & 1/10 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/10 & 1/2 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/10 & 1/10 & 1/2 & 1/10 \\ 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/2 \end{pmatrix} = \begin{pmatrix} x & y & y & y & y & y \end{pmatrix}$$

along with $x + 5y = 1$. Just computing first term in product gives $(5/6)x + 5 \cdot (1/10)y = x$ so that $x/6 = y/2$ and $x = 3y$. Using $x + 5y = 1$ we obtain $y = 1/8$ and $x = 3/8$.

2. (10 point) Harry is hosting a party when he spots a stylish cash-only burrito truck. In a moment of generosity he shouts, "Burritos on me!" He gets nervous when he sees the prices:

Sweet potato and black bean burrito: \$17
Chipotle shrimp and feta burrito: \$19
Teriyaki tofu and edamame burrito: \$20
Sweet potato and black bean burrito with guacamole: \$21
Chipotle shrimp and feta burrito with guacamole: \$23

Harry knows that each of his 25 friends will (independently of the others) order each of the five burrito types with equal likelihood.

(a) Let A_i be the price of the ith friend's burrito. Compute the mean and variance of A_1. **ANSWER:** $E[A_1] = 20$ and $\text{Var}(A_1) = E[(A_1 - 20)^2] = \frac{1}{5}(9 + 1 + 0 + 1 + 9) = 4$.

(b) Compute the mean and variance of the total price $A = A_1 + A_2 + \ldots + A_{25}$. **ANSWER:** A_i independent, so variance additivity gives $\text{Var}(A) = 25 \cdot 4 = 100$. Additivity of expectation gives $E[A] = 25 \cdot 20 = 500$

(c) Use a normal approximation to estimate the probability that the \$510 in Harry's wallet will cover the bill. You can use the function $\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **ANSWER:** \$510 is one standard deviation above mean, so answer is about $\phi(1)$.

(d) Estimate the probability that Harry's cash will cover the burritos for his friends *and* a teriyaki tofu and edamame burrito for himself. **ANSWER:** \$490 is one standard deviation below mean, so answer is about $\phi(-1)$.

3. (10 points) A college dorm has 8 floors. Each floor can house 12 students. However, there are only 9 students living on each floor now (for a total of $8 \times 9 = 72$ students). Dorm Supervisor Alice says, "Instead of 8 partially full floors, we should have 6 *full* floors and 2 *empty* floors. Then we can turn the empty floors into epic banana lounges." Alice devises a way to make that happen. Each day she will choose a pair of floors at random, uniformly from the set of $\binom{8}{2} = 28$ pairs. If each of the two floors has between 1 and 11 students, Alice will pick one of the two floors (with a fair coin toss) and transfer a student from that floor to the other floor. If either floor has 0 or 12 students, nothing is done on that day.

Let A_n be the number of students on Floor 1 after n days. (So $A_0 = 9$ with probability 1, but A_1 could be 8, 9 or 10.) Let K_n be the number of times a student is moved into or out of Floor 1 during the first n days. (So $K_0 = 0$ with probability 1, but K_1 could be 1 or 0.)

(a) Which of the following is a martingale? (Just circle the corresponding numbers.)

(i) $12A_n$ **YES:** When change can occur, coin toss makes shifting up and down 1 equally likely.

(ii) $12K_n$ **NO:** K_n can go up but never down.

(iii) $A_n + 12K_n$ **NO:** A_n is martingale but second term goes up in expectation

(iv) $A_n^2 - K_n$ **YES:** When change can occur, this becomes $(A_n + 1)^2 - (K_n + 1)$ or $(A_n - 1)^2 - (K_n + 1)$ with equal probability. Average of those two possibilities is $A_n^2 - K_n$.

(v) $12A_n - A_n^2 + K_n$ **YES:** This is (i) minus (iv). Difference of martingales is a martingale.

(vi) 54 **YES:** Constant function is a martingale.

(vii) $\sum_{i=1}^{n} A_i$ **NO:** This can go up but not down. Expected change not zero.

(b) Let T be the first time at which A_n reaches 0 or 12. Compute the expectation $E[K_T]$. **Hint:** Use one of the above martingales. **ANSWER:** Let M_n be the martingale (v). Note that A_T is 0 or 12 so $M_T = K_T$ and $E[M_T] = M_0 = 12 \cdot 9 - 81 = 27$.

(c) Let S be the first time at which *all* floors contain 0 or 12 students. Let N be the *total number of days* on which a move has occurred (between any pair of floors) by time S. Compute the expectation $E[N]$. **Hint:** you can use additivity of expectation but be careful about double counting. **ANSWER:** Sum answer from (b) over 8 floors (and divide by two, since each move involves two floors) to get $4 \cdot 27 = 108$.

4. (10 points) Bonnie plans to randomly select holiday gifts for her 10 grandchildren. She will give each grandchild

(i) a Walmart gift card with probability 1/2
(ii) a box of chocolate-covered cinnamon bears with probability 1/4
(iii) an internet-enabled hairbrush with probability 1/8
(iv) a pair of hand-knit wool socks with probability 1/32
(v) a vintage Bruce Springsteen record with probability 1/32
(vi) a front row ticket to *Renaissance: A Film by Beyoncé* with probability 1/32
(vii) a DVD recording of *The Princess Bride* with probability 1/32

and the choices will be independent from one child to the next. Let G_i be the type of gift given to the ith grandchild. Write G for the ordered list $(G_1, G_2, \dots, G_{10})$.

(a) Compute the entropy $H(G_1)$ and $H(G)$. **ANSWER:**
$H(G_1) = \frac{1}{2}(-\log\frac{1}{2}) + \frac{1}{4}(-\log\frac{1}{4}) + \frac{1}{8}(-\log\frac{1}{8}) + 4 \cdot \frac{1}{32}(-\log\frac{1}{32}) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 5 = 2$.
Since the components of G are independent we have $H(G) = 10 \cdot 2 = 20$.

(b) Suppose you want to determine G with a sequence of yes or no questions. What strategy minimizes the expected number of questions you have to ask? What is the expected number of questions needed in this case? **ANSWER:** For each child, ask: Is it (i)? If not, is it (ii)? If not, is it (iii)? If not, use two questions to decide between other 4 options. Takes $2 \cdot 10 = 20 = H(G)$ questions on average, which is the best possible.

(c) What is the probability that exactly one grandchild gets a Springsteen record? **ANSWER:** Number of Springsteen records is binomial with $n = 10$, $p = 1/32$. Answer: $\binom{10}{1}(1/32)^1(31/32)^9$.

5. (10 points) Let X be an exponential random variable with parameter $\lambda = 1$. For each real number K write $C(K) = E[\max\{X - K, 0\}]$.

(a) Compute $C(K)$ as a function of K for $K \geq 0$. **ANSWER:** $C(K) = \int_K^\infty e^{-x}(x-K)dx$. Integrate by part. Or write $y = x - K$, find $\int_0^\infty e^{-(y+K)}ydy = e^{-K}\int e^{-y}ydy = e^{-K}$.

(b) Compute the derivatives C' and C'' on $[0, \infty)$. **ANSWER:** $C'(K) = -e^{-K}$ and $C''(K) = e^{-K}$. Can also get this by recalling $C'(K) = F_X(K) - 1 = -P(X > K)$ and $C''(K) = f_X(K)$.

(c) Compute the expectation $E[X^3 + 3X^2 + 3X + 1]$. **ANSWER:** $3! + 3 \cdot 2! + 3 \cdot 1! + 1 = 16$.

6. (10 points) An exam has 25 problems, all equally difficult. Let T_i be the score that Terry the Test Taker earns on the ith problem. Assume that $T_1, T_2, \dots, T_{25}$ are i.i.d. random variables with mean μ and variance σ^2. Let $T = \sum_{i=1}^{25} T_i$ be Terry's total test score.

(a) Compute the correlation coefficient $\rho(T_1, T)$. **ANSWER:**
$\mathrm{Cov}(T_1, T_1 + T_2 + \ldots + T_{25}) = \mathrm{Cov}(T_1, T_1) + \mathrm{Cov}(T_1, T_2) + \ldots + \mathrm{Cov}(T_1, T_{25})$. First term is σ^2, others 0 by independence. Find $\rho(T_1, T) = \frac{\mathrm{Cov}(T_1,T)}{\sqrt{\mathrm{Var}(T_1)\mathrm{Var}(T_{25})}} = \frac{\sigma^2}{\sqrt{\sigma^2 \cdot 25\sigma^2}} = \frac{\sigma^2}{5\sigma^2} = \frac{1}{5}$.

(b) Express the conditional expectation $E[T_1|T]$ as a function of T. **ANSWER:** We know $T = E[T|T] = E[T_1 + T_2 + \ldots + T_{25}|T] = 25E[T_1|T]$ by symmetry. So $E[T_1|T] = \frac{1}{25}E[T]$.

(c) Compute the covariance $\mathrm{Cov}(T_1 + 2T_2 + 3T_3 + 9, T_1 + T_2 + T_3 + 7)$. **ANSWER:** By bilinearity of covariance, get sum over 16 terms of form $\mathrm{Cov}(A, B)$ where A is term from first sum, B term from second. Non-zero terms: $\mathrm{Cov}(T_1, T_1) + \mathrm{Cov}(2T_2, T_2) + \mathrm{Cov}(3T_3, T_3) = \sigma^2 + 2\sigma^2 + 3\sigma^2 = 6\sigma^2$.

(d) If we are given that T is R standard deviations above *its* mean, then how many standard deviations do we expect T_1 to be above *its* mean? In other words, write $R = \frac{T-E[T]}{\mathrm{SD}(T)}$ and $R_1 = \frac{T_1-E[T_1]}{\mathrm{SD}(T_1)}$ and express the random variable $E[R_1|R]$ in terms of R. **Hint:** The answer comes out to be a constant times R. **ANSWER:** First recall that by additivity of expectation and variance, we have $E[T] = 25E[T_1]$ and $\mathrm{Var}(T) = 25\mathrm{Var}(T_1)$ so that $\mathrm{SD}(T) = 5\mathrm{SD}(T_1)$. Next, note that R encodes same information as T so $E[R_1|R] = E[R_1|T] = \frac{T/25-E[T_1]}{\mathrm{SD}(T_1)} = \frac{(T-E[T])/25}{\mathrm{SD}(T)/5} = R/5$.

7. (10 points) Suppose that the pair (X, Y) has joint probability density function $f(x, y) = \frac{1}{\pi(1+x^2)} \cdot \frac{1}{\pi(1+y^2)}$.

(a) Compute the probability that the pair (X, Y) belongs to the unit box $[0, 1] \times [0, 1]$. **ANSWER:** 1/16 since X, Y independent Cauchy and (spinning flashlight) each has 1/4 chance to be in $[0, 1]$.

(b) Compute the probability that the pair (X, Y) belongs to the infinite diagonal strip $S = \{(x, y) : -2 < x + y < 2\}$. **ANSWER:** $Z = (X + Y)/2$ is Cauchy, so integral is $P\big(X + Y \in (-2, 2)\big) = P\big(Z \in (-1, 1)\big) = 1/2$.

(c) Express the expectation $E[\sin(XY)]$ as a double integral: you don't have to explicitly evaluate the integral. **ANSWER:**
$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \cdot \frac{1}{\pi(1+y^2)} \sin(xy) dxdy$

8. (10 points) Carol is a competitive basketball player, but she is prone to verbal outbursts. At successive times $A_1, A_2, \ldots$ she shouts, "Argh!" At times $B_1, B_2, \ldots$ she shouts, "Blimey!" and at times $C_1, C_2, \ldots$ she shouts "Curses!" Assume that the times A_i, the times B_i and the times C_i are independent Poisson point processes with respective intensities $\lambda_A = 1$/hour, $\lambda_B = 2$/hour and $\lambda_C = 3$/hour.

(a) Compute the probability that, during 2 hours of play, Carol uses "argh," "blimey" and "curses" exactly 4 times each. **ANSWER:** $e^{-2\cdot 1}\frac{(2\cdot 1)^4}{4!} \cdot e^{-2\cdot 2}\frac{(2\cdot 2)^4}{4!} \cdot e^{-2\cdot 3}\frac{(2\cdot 3)^4}{4!} = e^{-12}(2^4 \cdot 4^4 \cdot 6^4)/(4!)^3$.

(b) Write down the density function for Carol's first vocal outburst. In other words, write f_X where $X = \min\{A_1, B_1, C_1\}$. **ANSWER:** This is exponential with parameter $\lambda_A + \lambda_B + \lambda_C = 6$. So $f_X(x) = 6e^{-6x}$ on $[0, \infty)$.

(c) Write the covariance $\mathrm{Cov}(A_1 + B_2 + C_3, A_3 + B_2 + C_1)$. **ANSWER:** Bilinearity gives sum over 9 terms of form $\mathrm{Cov}(R, S)$ where R comes from first sum, S from second. Nonzero terms are $\mathrm{Cov}(A_1, A_3) + \mathrm{Cov}(B_2, B_2) + \mathrm{Cov}(C_3, C_1)$. A_3 is sum of three independent increments, and bilinearity of covariance gives $\mathrm{Cov}(A_1, A_3) = \mathrm{Cov}(A_1, A_1) = \mathrm{Var}(A_1)$. Similarly $\mathrm{Cov}(C_3, C_1) = \mathrm{Var}(C_1)$. And B_2 is sum of 2 independent increments: additivity of variance gives $\mathrm{Var}(B_2) = 2\mathrm{Var}(B_1)$. Answer: $\mathrm{Var}(A_1) + 2\mathrm{Var}(B_2) + \mathrm{Var}(C_1) = 1 + 2 \cdot (1/2)^2 + (1/3)^2 = 29/18$.

(d) The coach decides to withdraw Carol from play after the third time she says "curses" (there are children present after all). What is the probability density function for the amount of play (in hours) until this happens? In other words, what is f_{C_3}? **ANSWER:** This is Gamma with parameter 3 and 3. So the density function is $\frac{3e^{-(3x)}(3x)^2}{2!}$ on $[0,\infty)$.

9. (10 points) Ten friends remove their shoes before entering a Japanese restaurant. When they leave, they are each given a pair of shoes—one left shoe and one right shoe. However, the left shoes have been randomly permuted (all 10! permutations equally likely) and the right shoes have been independently randomly permuted (all 10! permutations equally likely). Let A be the number of people who get *both* of their own shoes back.

(a) Compute the expectation $E[A]$. **Hint:** If it helps, you can write A_i for the random variable that is 1 if the ith person gets both of their own shoes back, 0 otherwise. **ANSWER:** By additivity of expectation $E[A] = E[\sum_{i=1}^{10} A_i] = \sum_{i=1}^{10} E[A_i] = 10E[A_1] = 10 \cdot (1/100) = 1/10$.

(b) Compute the expectation $E[A^2]$. **ANSWER:** $E[A] = E[\sum_{i=1}^{10}\sum_{j=1}^{10} A_iA_j] = \sum_{i=1}^{10}\sum_{j=1}^{10} E[A_iA_j]$. There are 10 times with $i = j$ and 90 with $i \neq j$. Answer is $10E[A_1A_1] + 90E[A_1A_2] = 10 \cdot \frac{1}{100} + 90 \cdot (\frac{1}{90})^2 = 1/10 + 1/90 = 1/9$.

(c) Let B be the number of people who are given a left shoe and a right shoe that *match one another* (i.e. two shoes that originally had the same owner). Compute $E[AB]$. **ANSWER:** $E[\sum A_i \sum B_j] = \sum\sum E[A_iB_j]$. There are ten terms equal to $E[A_iB_i] = E[A_i] = 1/100$. There are 90 terms equal to $E[A_1B_2] = (1/100)(1/9) = 1/900$. So answer is $10/100 + 90/900 = 2/10$.

10. (10 points) At the end of a long semester, after 300 students take a final exam, the graders decide to save time by assigning grades randomly. They sample $X_1, X_2, \ldots X_{300}$ as i.i.d. uniform random variables on $[0,1]$ and assign the ith student the score X_i. Write $S = \sum_{i=1}^{300} X_i$ for the total of the scores and $A = S/300$ for the average.

(a) Compute the moment generating function $M_{X_1}(t)$. **ANSWER:** $E[e^{tX_1}] = \int_0^1 1 \cdot e^{tx}dx = \frac{e^t}{t} - \frac{e^0}{t} = \frac{e^t-1}{t}$

(b) Compute the moment generating functions $M_S(t)$ and $M_A(t)$. **ANSWER:** $M_S(t) = \prod_{i=1}^{300} = M_{X_i}(t) = \left(\frac{e^t-1}{t}\right)^{300}$. Then $M_A(t) = M_{S/300}(t) = M_S(t/300) = \left(\frac{e^{t/300}-1}{(t/300)}\right)^{300}$

(c) Let L be the largest of the X_i and compute the density function f_L. **ANSWER:** $F_L(a) = P(L \leq a) = a^{300}$ so $f_L(a) = 300a^{299}$ on $[0,1]$.

(d) Compute the standard deviation of S. Use the central limit theorem to approximate the probability that $S > 165$ (so that the class average A is higher than 55 percent). You can use the function $\Phi(a) := \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$ in your answer. **ANSWER:** Each X_i has variance 1/12 so $\mathrm{Var}(S) = 25$ and $\mathrm{SD}(S) = 5$. Also $E[S] = 150$ so 165 is 3 SDs above that. The chance over being more than three SDs above mean is about $1 - \Phi(3) = \Phi(-3)$ which is pretty small.