18.600 Midterm 1, Fall 2023: 50 minutes, 100 points

1. Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.

2. No calculators, books, or notes may be used.
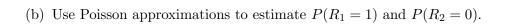
3. Simplify your answers as much as possible (but answers may include factorials and $\binom{n}{k}$ expressions — no need to multiply them out).

NAME: _____

1. (20 points) The online commentators on Opinion Planet routinely promote 20 specific opinions. (Opinion 1: Income tax should be optional. Opinion 2: We need more squirrels....) Alice decides to select 10 of the 20 to passionately support, 5 to passionately oppose, and 5 to be indifferent about.

(a) Let N be the total number of ways she can do that. Compute N.

(b) Alice randomly adopts one of the N possibilities (all N being equally likely). Bob is contrarian. He independently chooses 12 of the 20 opinions to strongly oppose (all sets of 12 equally likely) and stays indifferent on the rest. Let D be the number of *passionate disagreements* (i.e., opinions that Alice passionately supports *and* Bob passionately opposes). Compute $E(D)$. **Note:** If it helps, you can define D_i to be 1 if they passionately disagree on the ith opinion, 0 otherwise.

(c) Compute the probability p that Alice and Bob passionately disagree on *both* Opinions 1 and 2.

(d) Compute the expectation $E(D^2)$. If it helps, you can express your answer in terms of the p from part (c). (No need to plug in and simplify.)

2. (15 points) Ten bankers gather at a Swiss retreat. The first has a watch worth $1000, the second has a watch worth $2000, and generally the nth banker has a watch worth $1000n$. They remove their watches to use the spa. Each banker retrieves a watch afterward, but since the watches appear identical at first glance, they get all mixed up, with each of the 10! permutations being equally likely.

(a) Compute the expected number of bankers who end up with a watch that is *more valuable* than the one they started out with.

(b) As a function of $k \in \{1, 2, \ldots, 10\}$, find the probability p_k that *all* of the first k bankers get their own watches.

(c) Find the probability that at least one *of the first four* bankers gets their own watch back. If it helps, you can express your answer in terms of the p_k from (b). (No need to plug in and simplify.)

3. (20 points) Bob prepares two online dating photos: Photo 1 (pensively strumming a guitar) and Photo 2 (tenderly cuddling a puppy). His dating service sends Photo 1 to 1000 potential partners, each of whom responds (independently of all else) with probability $1/500$. The service sends Photo 2 to 1000 different people, each of whom responds (independently of all else) with probability $1/250$. Let R_1 and R_2 be the number of responses to Photo 1 and Photo 2, respectively.

(a) Compute $P(R_1 = k)$ for $k \in \{0, 1, \ldots, 1000\}$. Given an exact formula, not a Poisson approximation.

(b) Use Poisson approximations to estimate $P(R_1 = 1)$ and $P(R_2 = 0)$.

(c) Use Poisson approximations to estimate the probability that Bob receives one response total. That is, estimate $P(R_1 + R_2 = 1)$.

(d) Use Poisson approximations to estimate the *conditional probability* that the response is from somebody who viewed the puppy picture, *given* that Bob received exactly one response overall. In other words, estimate $P(R_2 = 1 | R_1 + R_2 = 1)$. Express your answer as a simple rational number.

4. (10 points) Six teenagers trick-or-treating late at night encounter a bucket containing 30 candy bars and a sign that reads "Help yourself!" Let c_i represent the number of candy bars the ith teenager takes.

(a) How many 6-tuples $(c_1, \ldots, c_6)$ are possible? In other words, how many ordered 6-tuples $(c_1, c_2, \ldots, c_6)$ of *non-negative* integers satisfy $c_1 + c_2 + \ldots + c_6 \leq 30$? (Caution: don't mistake the $\leq$ for an equality. Not all bars have to be taken.)

(b) Suppose the teenagers decide to take all the candy bars but that (to be fair) they will independently roll a 6-sided die for each candy bar to decide which person gets it. Let $A_1, A_2, \ldots, A_{30}$ be the numbers that come up on the 30 successive die rolls. (So each A_i is a random integer in $\{1, 2, \ldots, 6\}$.) The ith bar goes to the A_ith teenager. What is the probability that the candy bars end up evenly divided (i.e. the probability that each number between 1 and 6 appears in the sequence $A_1, \ldots, A_{30}$ exactly five times)?

5.(15 points) Compute the following:

(a) $\displaystyle\sum_{k=0}^{6} \binom{6}{k} 9^k$

(b) $\displaystyle\lim_{n\to\infty} \left(1 - \frac{3}{2n}\right)^n$

(c) $\displaystyle\sum_{k=0}^{\infty} e^{-5}\left(\frac{5^k}{k!} \cdot k^2\right)$

6. (10 points) How many "two pair" poker hands are there? **Recall:** there are 13 face values (2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q, A). A deck has 4 distinct cards of each face value. A "two pair" hand consists of 5 cards from the same deck, with two cards of one face value, two cards of a second (distinct) face value, and one card of a third (distinct) face value.

7.(10 points) Let X be a random variable with $P(X = -2) = 1/4$ and $P(X = 2) = 3/4$. Compute:

(a) $\text{Var}(X)$

(b) $\text{Var}(19 + 20X + 21X^2 + 20X^3 + 19X^4)$