18.600: Lecture 36-37 Review: practice problems

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Eight athletic teams are ranked 1 through 8 after season one, and ranked 1 through 8 again after season two. Assume that each set of rankings is chosen uniformly from the set of 8! possible rankings and that the two rankings are independent. Let N be the number of teams whose rank does not change from season one to season two. Let N₊ the number of teams whose rank improves by exactly two spots. Let N₋ be the number whose rank declines by exactly two spots. Compute the following: Eight athletic teams are ranked 1 through 8 after season one, and ranked 1 through 8 again after season two. Assume that each set of rankings is chosen uniformly from the set of 8! possible rankings and that the two rankings are independent. Let N be the number of teams whose rank does not change from season one to season two. Let N₊ the number of teams whose rank improves by exactly two spots. Let N₋ be the number whose rank declines by exactly two spots. Compute the following:

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 - \blacktriangleright E[N], E[N₊], and E[N₋]
 - ► Var[*N*]
 - ► $Var[N_+]$

▶ Let N_i be 1 if team ranked *i*th first season remains *i*th second seasons. Then $E[N] = E[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1$. Similarly, $E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4$

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 Var[N] = E[N²] - E[N]² and E[N²] = E[∑_{i=1}⁸ ∑_{i=1}⁸ N_iN_i] = 8 ⋅ ¹/₈ + 56 ⋅ ¹/₅₆ = 2.

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►
$$\operatorname{Var}[N] = E[N^2] - E[N]^2$$
 and
 $E[N^2] = E[\sum_{i=1}^8 \sum_{j=1}^8 N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2.$

▶ N_{+}^{i} be 1 if team ranked *i*th has rank improve to (i - 2)th for second seasons. Then $E[(N_{+})^{2}] = E[\sum_{j=1}^{8} \sum_{3=1}^{8} N_{+}^{i} N_{+}^{j}] = 6 \cdot \frac{1}{8} + 30 \cdot \frac{1}{56} = 9/7$, so $Var[N_{+}] = 9/7 - (3/4)^{2}$. Roll ten dice. Find the conditional probability that there are exactly 4 ones, given that there are exactly 4 sixes. Straightforward approach: P(A|B) = P(AB)/P(B).

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 ▶ Numerator: is ⁽¹⁰⁾₄(⁶)₄)⁴²/_{6¹⁰}. Denominator is ⁽¹⁰⁾_{6¹⁰}5⁶/_{6¹⁰}.

Straightforward approach: P(A|B) = P(AB)/P(B).
 Numerator: is (¹⁰₄)(⁶₄)4²/_{6¹⁰}. Denominator is (¹⁰₄)5⁶/_{6¹⁰}.
 Ratio is (⁶₄)4²/5⁶ = (⁶₄)(¹/₅)⁴(⁴/₅)².

- Straightforward approach: P(A|B) = P(AB)/P(B).
- Numerator: is $\frac{\binom{10}{4}\binom{6}{4}4^2}{6^{10}}$. Denominator is $\frac{\binom{10}{4}5^6}{6^{10}}$.
- Ratio is $\binom{6}{4}4^2/5^6 = \binom{6}{4}(\frac{1}{5})^4(\frac{4}{5})^2$.
- Alternate solution: first condition on location of the 6's and then use binomial theorem.

- Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. The V be length of time (in decades) until the first volcano eruption and E the length of time (in decades) until the first earthquake. Compute the following:
 - ▶ $\mathbb{E}[E^2]$ and $\operatorname{Cov}[E, V]$.

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 - $\blacktriangleright \mathbb{E}[E^2] \text{ and } \operatorname{Cov}[E, V].$
 - The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.

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 - The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
 - ► The probability density function of min{*E*, *V*}.

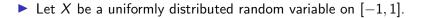
•
$$E[E^2] = 2$$
 and $Cov[E, V] = 0$.

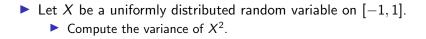
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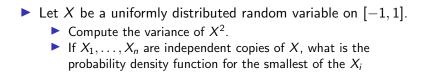
Probability of no earthquake or eruption in first year is e^{-(2+1)¹/₁₀} = e^{-.3} (see next part). Same for any year by memoryless property. Expected number of quake/eruption-free years is 10e^{-.3} ≈ 7.4.

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- Probability density function of min{E, V} is 3e^{-(2+1)x} for x ≥ 0, and 0 for x < 0.</p>







Order statistics — answers

 $Var[X^2] = E[X^4] - (E[X^2])^2$ $= \int_{-1}^{1} \frac{1}{2} x^4 dx - (\int_{-1}^{1} \frac{1}{2} x^2 dx)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$

Order statistics — answers

 $Var[X^2] = E[X^4] - (E[X^2])^2$ $= \int_{-1}^{1} \frac{1}{2} x^4 dx - (\int_{-1}^{1} \frac{1}{2} x^2 dx)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$ Note that for $x \in [-1, 1]$ we have $P\{X > x\} = \int_{-1}^{1} \frac{1}{2} dx = \frac{1-x}{2}.$ If $x \in [-1, 1]$, then $P\{\min\{X_1, \ldots, X_n\} > x\}$ $= P\{X_1 > x, X_2 > x, \dots, X_n > x\} = (\frac{1-x}{2})^n.$

So the density function is

$$-\frac{\partial}{\partial x}\left(\frac{1-x}{2}\right)^n = \frac{n}{2}\left(\frac{1-x}{2}\right)^{n-1}.$$

Suppose that X_i are independent copies of a random variable X. Let M_X(t) be the moment generating function for X. Compute the moment generating function for the average ∑ⁿ_{i=1} X_i/n in terms of M_X(t) and n.

• Write
$$Y = \sum_{i=1}^{n} X_i / n$$
. Then
 $M_Y(t) = E[e^{tY}] = E[e^{t\sum_{i=1}^{n} X_i / n}] = (M_X(t/n))^n$.

Suppose X and Y are independent random variables, each equal to 1 with probability 1/3 and equal to 2 with probability 2/3.

• Compute the entropy H(X).

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 - Compute the entropy H(X).
 - Compute H(X + Y).
 - Which is larger, H(X + Y) or H(X, Y)? Would the answer to this question be the same for any discrete random variables X and Y? Explain.

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• $H(X+Y) = \frac{1}{9}(-\log \frac{1}{9}) + \frac{4}{9}(-\log \frac{4}{9}) + \frac{4}{9}(-\log \frac{4}{9})$

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- After Bob showers, if there is at least one towel in the bathroom, Bob uses the towel and leaves it draped over a chair in the walk-in closet. If there is no towel in the bathroom, Bob grumpily goes to the walk-in closet, dries off there, and leaves the towel in the walk-in closet

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- When Alice showers, she first checks to see if at least one towel is present. If a towel is present, she dries off with that towel and returns it to the bathroom towel rack. Otherwise, she cheerfully retrieves both towels from the walk-in closet, then showers, dries off and leaves both towels on the rack.

Markov chains

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- Problem: describe towel-distribution evolution as a Markov chain and determine (over the long term) on what fraction of days Bob emerges from the shower to find no towel.

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- Morning state change AB: $2 \rightarrow 1$, $1 \rightarrow 0$, $0 \rightarrow 1$.

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- Markov chain matrix:

$$M = \begin{pmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{pmatrix}$$

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• Row vector π such that $\pi M = \pi$ (with components of π summing to one) is $\begin{pmatrix} 2 \\ 9 \\ 4 \\ 9 \\ 1 \\ 3 \end{pmatrix}$.

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- Row vector π such that $\pi M = \pi$ (with components of π summing to one) is $\begin{pmatrix} 2 \\ 9 & 4 \\ 9 & \frac{1}{3} \end{pmatrix}$.
- Bob finds no towel only if morning starts in state zero and Bob goes first. Over long term Bob finds no towel ²/₉ × ¹/₂ = ¹/₉ fraction of the time.

Suppose that $X_1, X_2, X_3, ...$ is an infinite sequence of independent random variables which are each equal to 1 with probability 1/2 and -1 with probability 1/2. Let $Y_n = \sum_{i=1}^n X_i$. Answer the following:

▶ What is the probability that Y_n reaches −25 before the first time that it reaches 5?

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- ▶ What is the probability that Y_n reaches −25 before the first time that it reaches 5?
- Use the central limit theorem to approximate the probability that $Y_{9000000}$ is greater than 6000.

Optional stopping, martingales, central limit theorem — answers

▶ $p_{-25}25 + p_55 = 0$ and $p_{-25} + p_5 = 1$. Solving, we obtain $p_{-25} = 1/6$ and $p_5 = 5/6$.

Optional stopping, martingales, central limit theorem — answers

- ▶ $p_{-25}25 + p_55 = 0$ and $p_{-25} + p_5 = 1$. Solving, we obtain $p_{-25} = 1/6$ and $p_5 = 5/6$.
- One standard deviation is $\sqrt{9000000} = 3000$. We want probability to be 2 standard deviations above mean. Should be about $\int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Let X_i be independent random variables with mean zero. In which of the cases below is the sequence Y_i necessarily a martingale?

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$$Y_n = \sum_{i=1}^n iX_i Y_n = \sum_{i=1}^n X_i^2 - n$$

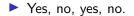
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UNDERGRADUATE:

- (a) 18.615 Introduction to Stochastic Processes
- (b) 18.642 Topics in Math with Applications in Finance
- (c) 18.650 Statistics for Applications

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- (b) 18.676 Stochastic calculus
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- OUTSIDE OF MATH DEPARTMENT
 - (a) Look up new MIT minor in statistics and data sciences.
 - (b) Look up longer lists of probability/statistics courses at https: //stat.mit.edu/academics/minor-in-statistics/ or http://student.mit.edu/catalog/m18b.html
 - (c) Ask other MIT faculty how they use probability and statistics in their research.

 Considering previous generations of mathematically inclined MIT students, and adopting a frequentist point of view...

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- And may the odds be ever in your favor.