18.600: Lecture 36-37

Review: practice problems

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Expectation and variance

- Eight athletic teams are ranked 1 through 8 after season one, and ranked 1 through 8 again after season two. Assume that each set of rankings is chosen uniformly from the set of 8! possible rankings and that the two rankings are independent. Let N be the number of teams whose rank does not change from season one to season two. Let N_+ the number of teams whose rank improves by exactly two spots. Let N_- be the number whose rank declines by exactly two spots. Compute the following:

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Expectation and variance — answers

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- $\mathrm{Var}[N] = E[N^2] - E[N]^2$ and $E[N^2] = E[\sum_{i=1}^{8} \sum_{j=1}^{8} N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2.$

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- N_+^i be 1 if team ranked ith has rank improve to $(i-2)$th for second seasons. Then $E[(N_+)^2] = E[\sum_{j=1}^{8} \sum_{3=1}^{8} N_+^i N_+^j] = 6 \cdot \frac{1}{8} + 30 \cdot \frac{1}{56} = 9/7$, so $\mathrm{Var}[N_+] = 9/7 - (3/4)^2$.

Conditional distributions

- Roll ten dice. Find the conditional probability that there are exactly 4 ones, given that there are exactly 4 sixes.

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- Alternate solution: first condition on location of the 6's and then use binomial theorem.

Poisson point processes

- Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. The V be length of time (in decades) until the first volcano eruption and E the length of time (in decades) until the first earthquake. Compute the following:
  - $\mathbb{E}[E^2]$ and $\mathrm{Cov}[E, V]$.

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  - $\mathbb{E}[E^2]$ and $\mathrm{Cov}[E, V]$.
  - The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
  - The probability density function of $\min\{E, V\}$.

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- Probability of no earthquake or eruption in first year is $e^{-(2+1)\frac{1}{10}} = e^{-.3}$ (see next part). Same for any year by memoryless property. Expected number of quake/eruption-free years is $10e^{-.3} \approx 7.4$.

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- Probability density function of $\min\{E, V\}$ is $3e^{-(2+1)x}$ for $x \geq 0$, and 0 for $x < 0$.

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  - If $X_1, \ldots, X_n$ are independent copies of X, what is the probability density function for the smallest of the X_i

Order statistics — answers

- $$\mathrm{Var}[X^2] = E[X^4] - (E[X^2])^2$$
$$= \int_{-1}^{1} \frac{1}{2} x^4 dx - (\int_{-1}^{1} \frac{1}{2} x^2 dx)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$$

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- Note that for $x \in [-1, 1]$ we have
  $$P\{X > x\} = \int_x^1 \frac{1}{2} dx = \frac{1-x}{2}.$$
  If $x \in [-1, 1]$, then
  $$P\{\min\{X_1, \ldots, X_n\} > x\}$$
  $$= P\{X_1 > x, X_2 > x, \ldots, X_n > x\} = (\frac{1-x}{2})^n.$$
  So the density function is
  $$-\frac{\partial}{\partial x}(\frac{1-x}{2})^n = \frac{n}{2}(\frac{1-x}{2})^{n-1}.$$

Moment generating functions

- Suppose that X_i are independent copies of a random variable X. Let $M_X(t)$ be the moment generating function for X. Compute the moment generating function for the average $\sum_{i=1}^n X_i/n$ in terms of $M_X(t)$ and n.

Moment generating functions — answers

- Write $Y = \sum_{i=1}^{n} X_i/n$. Then
$$M_Y(t) = E[e^{tY}] = E[e^{t\sum_{i=1}^{n} X_i/n}] = (M_X(t/n))^n.$$

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  - Compute $H(X+Y)$.
  - Which is larger, $H(X+Y)$ or $H(X,Y)$? Would the answer to this question be the same for any discrete random variables X and Y? Explain.

Entropy — answers

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- $H(X,Y)$ is larger, and we have $H(X,Y) \geq H(X+Y)$ for any X and Y. To see why, write $a(x,y) = P\{X = x, Y = y\}$ and $b(x,y) = P\{X + Y = x + y\}$. Then $a(x,y) \leq b(x,y)$ for any x and y, so
  $H(X,Y) = E[-\log a(x,y)] \geq E[-\log b(x,y)] = H(X+Y)$.

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- After Bob showers, if there is at least one towel in the bathroom, Bob uses the towel and leaves it draped over a chair in the walk-in closet. If there is no towel in the bathroom, Bob grumpily goes to the walk-in closet, dries off there, and leaves the towel in the walk-in closet

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- When Alice showers, she first checks to see if at least one towel is present. If a towel is present, she dries off with that towel and returns it to the bathroom towel rack. Otherwise, she cheerfully retrieves both towels from the walk-in closet, then showers, dries off and leaves both towels on the rack.

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- **Problem:** describe towel-distribution evolution as a Markov chain and determine (over the long term) on what fraction of days Bob emerges from the shower to find no towel.

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- Markov chain matrix:

$$M = \begin{pmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{pmatrix}$$

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- Row vector π such that $\pi M = \pi$ (with components of π summing to one) is $\begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$.

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- Bob finds no towel only if morning starts in state zero and Bob goes first. Over long term Bob finds no towel $\frac{2}{9} \times \frac{1}{2} = \frac{1}{9}$ fraction of the time.

Optional stopping, martingales, central limit theorem

Suppose that $X_1, X_2, X_3, \dots$ is an infinite sequence of independent random variables which are each equal to 1 with probability $1/2$ and -1 with probability $1/2$. Let $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

- What is the the probability that Y_n reaches -25 before the first time that it reaches 5?

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Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 1 with probability $1/2$ and -1 with probability $1/2$. Let $Y_n = \sum_{i=1}^n X_i$. Answer the following:

- What is the the probability that Y_n reaches -25 before the first time that it reaches 5?
- Use the central limit theorem to approximate the probability that $Y_{9000000}$ is greater than 6000.

Optional stopping, martingales, central limit theorem — answers

- $p_{-25}25 + p_5 5 = 0$ and $p_{-25} + p_5 = 1$. Solving, we obtain $p_{-25} = 1/6$ and $p_5 = 5/6$.

Optional stopping, martingales, central limit theorem — answers

- $p_{-25}25 + p_5 5 = 0$ and $p_{-25} + p_5 = 1$. Solving, we obtain $p_{-25} = 1/6$ and $p_5 = 5/6$.
- One standard deviation is $\sqrt{9000000} = 3000$. We want probability to be 2 standard deviations above mean. Should be about $\int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

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Martingales

- Yes, no, yes, no.

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- **OUTSIDE OF MATH DEPARTMENT**
  (a) Look up new MIT minor in statistics and data sciences.
  (b) Look up longer lists of probability/statistics courses at `https://stat.mit.edu/academics/minor-in-statistics/` or `http://student.mit.edu/catalog/m18b.html`
  (c) Ask other MIT faculty how they use probability and statistics in their research.

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