18.600: Lecture 35

Martingales and risk neutral probability

Scott Sheffield

MIT

Outline

Martingales and stopping times

Risk neutral probability and martingales

Call function

Black-Scholes

Outline

Recall martingale definition

- Let S be the probability space. Let $X_0, X_1, X_2, \ldots$ be a sequence of real random variables. Interpret X_i as price of asset at ith time step.

Recall martingale definition

- Let S be the probability space. Let $X_0, X_1, X_2, \dots$ be a sequence of real random variables. Interpret X_i as price of asset at ith time step.
- Say X_n sequence is a **martingale** if $E[|X_n|] < \infty$ for all n and $E[X_{n+1}|\mathcal{F}_n] = X_n$ for all n.

Recall martingale definition

- Let S be the probability space. Let $X_0, X_1, X_2, \dots$ be a sequence of real random variables. Interpret X_i as price of asset at ith time step.
- Say X_n sequence is a **martingale** if $E[|X_n|] < \infty$ for all n and $E[X_{n+1}|\mathcal{F}_n] = X_n$ for all n.
- "Given all I know today, expected price tomorrow is the price today."

Recall stopping time definition

- Let T be a non-negative integer valued random variable.

Recall stopping time definition

- Let T be a non-negative integer valued random variable.
- Think of T as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.

Recall stopping time definition

- Let T be a non-negative integer valued random variable.
- Think of T as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.
- Say that T is a **stopping time** if the event that $T = n$ depends only on the values X_i for $i \leq n$. In other words, the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.

Recall stopping time definition

- Let T be a non-negative integer valued random variable.
- Think of T as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.
- Say that T is a **stopping time** if the event that $T = n$ depends only on the values X_i for $i \leq n$. In other words, the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.
- **Optional stopping theorem:—** As long as X_T is bounded (or T is bounded) between fixed constants with probability one, we have $E[X_T] = E[X_0]$.

Recall stopping time definition

- Let T be a non-negative integer valued random variable.
- Think of T as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.
- Say that T is a **stopping time** if the event that $T = n$ depends only on the values X_i for $i \leq n$. In other words, the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.
- **Optional stopping theorem:—** As long as X_T is bounded (or T is bounded) between fixed constants with probability one, we have $E[X_T] = E[X_0]$.
- **Informal proof if $P(T \leq N) = 1$ for some N:** OST says "If only care about expectation, selling at time 0 is as good as any strategy for selling at a time between 0 and N." If we make it to time $N-1$ then *at that point* we may as well sell, since martingale property implies $E[X_N|\mathcal{F}_{N-1}] = X_{N-1}$. Replace N by $N-1$, proceed inductively.

Outline

Martingales and stopping times

Risk neutral probability and martingales

Call function

Black-Scholes

Outline

Martingales applied to finance

- Many asset prices are believed to behave approximately like martingales, at least in the short term.

Martingales applied to finance

- Many asset prices are believed to behave approximately like martingales, at least in the short term.
- **Efficient market hypothesis**: new information is instantly absorbed into the stock value, so expected value of the stock tomorrow should be the value today. (If it were higher, statistical arbitrageurs would bid up today's price until this was not the case.)

Martingales applied to finance

- Many asset prices are believed to behave approximately like martingales, at least in the short term.
- **Efficient market hypothesis**: new information is instantly absorbed into the stock value, so expected value of the stock tomorrow should be the value today. (If it were higher, statistical arbitrageurs would bid up today's price until this was not the case.)
- But there are some caveats: interest, risk premium, bid-ask spread, etc.

Martingales applied to finance

- Many asset prices are believed to behave approximately like martingales, at least in the short term.
- **Efficient market hypothesis**: new information is instantly absorbed into the stock value, so expected value of the stock tomorrow should be the value today. (If it were higher, statistical arbitrageurs would bid up today's price until this was not the case.)
- But there are some caveats: interest, risk premium, bid-ask spread, etc.
- According to the **fundamental theorem of asset pricing**, the discounted price $\frac{X(n)}{A(n)}$, where A is a risk-free asset, is a martingale with respect to **risk neutral probability**.

Risk neutral probability

- "Risk neutral probability" is a fancy term for "market probability". (The term "market probability" is arguably more descriptive.)

Risk neutral probability

- “Risk neutral probability” is a fancy term for “market probability”. (The term “market probability” is arguably more descriptive.)
- That is, it is a probability measure that you can deduce by looking at prices on market.

Risk neutral probability

- "Risk neutral probability" is a fancy term for "market probability". (The term "market probability" is arguably more descriptive.)
- That is, it is a probability measure that you can deduce by looking at prices on market.
- For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?

Risk neutral probability

- "Risk neutral probability" is a fancy term for "market probability". (The term "market probability" is arguably more descriptive.)
- That is, it is a probability measure that you can deduce by looking at prices on market.
- For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?
- If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.

Risk neutral probability

- "Risk neutral probability" is a fancy term for "market probability". (The term "market probability" is arguably more descriptive.)
- That is, it is a probability measure that you can deduce by looking at prices on market.
- For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?
- If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.
- Risk neutral probability is the probability determined by the market betting odds.

Risk neutral probability

- "Risk neutral probability" is a fancy term for "market probability". (The term "market probability" is arguably more descriptive.)
- That is, it is a probability measure that you can deduce by looking at prices on market.
- For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?
- If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.
- Risk neutral probability is the probability determined by the market betting odds.
- Assume bid-ask spread essentially zero. (Otherwise risk neutral probability would be somewhere between bid and ask, wouldn't know where.)

Risk neutral probability of outcomes known at fixed time T

- **Risk neutral probability of event A:** $P_{RN}(A)$ denotes

$$\frac{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs }\}}{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ no matter what }\}}.$$

Risk neutral probability of outcomes known at fixed time T

- **Risk neutral probability of event A:** $P_{RN}(A)$ denotes

$$\frac{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs }\}}{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ no matter what }\}}.$$

- If risk-free interest rate is constant and equal to r (compounded continuously), then denominator is e^{-rT}.

Risk neutral probability of outcomes known at fixed time T

- **Risk neutral probability of event A:** $P_{RN}(A)$ denotes

$$\frac{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs }\}}{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ no matter what }\}}.$$

- If risk-free interest rate is constant and equal to r (compounded continuously), then denominator is e^{-rT}.
- Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment), P_{RN} satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events A_j are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$

Risk neutral probability of outcomes known at fixed time T

- **Risk neutral probability of event A:** $P_{RN}(A)$ denotes

$$\frac{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs }\}}{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ no matter what }\}}.$$

- If risk-free interest rate is constant and equal to r (compounded continuously), then denominator is e^{-rT}.
- Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment), P_{RN} satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events A_j are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$
- **Arbitrage example:** if A and B are disjoint and $P_{RN}(A \cup B) < P(A) + P(B)$ then we sell contracts paying 1 if A occurs and 1 if B occurs, buy contract paying 1 if $A \cup B$ occurs, pocket difference.

Risk neutral probability differ vs. "ordinary probability"

- At first sight, one might think that $P_{RN}(A)$ describes the market's best guess at the probability that A will occur.

Risk neutral probability differ vs. “ordinary probability”

- At first sight, one might think that $P_{RN}(A)$ describes the market’s best guess at the probability that A will occur.
- But suppose A is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?

Risk neutral probability differ vs. "ordinary probability"

- At first sight, one might think that $P_{RN}(A)$ describes the market's best guess at the probability that A will occur.
- But suppose A is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?
- Should be 0. Even if people think A is *likely*, a contract paying a dollar when A occurs is worthless.

Risk neutral probability differ vs. "ordinary probability"

- At first sight, one might think that $P_{RN}(A)$ describes the market's best guess at the probability that A will occur.
- But suppose A is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?
- Should be 0. Even if people think A is *likely*, a contract paying a dollar when A occurs is worthless.
- Now, suppose there are only 2 outcomes: A is event that economy booms and everyone prospers and B is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think A has a .5 chance to occur, do we expect $P_{RN}(A) > .5$ or $P_{RN}(A) < .5$?

Risk neutral probability differ vs. "ordinary probability"

- At first sight, one might think that $P_{RN}(A)$ describes the market's best guess at the probability that A will occur.
- But suppose A is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?
- Should be 0. Even if people think A is *likely*, a contract paying a dollar when A occurs is worthless.
- Now, suppose there are only 2 outcomes: A is event that economy booms and everyone prospers and B is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think A has a .5 chance to occur, do we expect $P_{RN}(A) > .5$ or $P_{RN}(A) < .5$?
- Answer: $P_{RN}(A) < .5$. People are risk averse. In second scenario they need the money more.

Non-systemic event

- Suppose that A is the event that the Boston Red Sox win the World Series. Would we expect $P_{RN}(A)$ to represent (the market's best assessment of) the probability that the Red Sox will win?

Non-systemic event

- Suppose that A is the event that the Boston Red Sox win the World Series. Would we expect $P_{RN}(A)$ to represent (the market's best assessment of) the probability that the Red Sox will win?
- Arguably yes. The amount that *people in general* need or value dollars does not depend much on whether A occurs (even though the financial needs of specific individuals may depend on heavily on A).

Non-systemic event

- Suppose that A is the event that the Boston Red Sox win the World Series. Would we expect $P_{RN}(A)$ to represent (the market's best assessment of) the probability that the Red Sox will win?
- Arguably yes. The amount that *people in general* need or value dollars does not depend much on whether A occurs (even though the financial needs of specific individuals may depend on heavily on A).
- Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to team's prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of what well-informed informed experts would consider the true probability.

Extensions of risk neutral probability

- Definition of risk neutral probability depends on choice of currency (the so-called *numéraire*).

Extensions of risk neutral probability

- Definition of risk neutral probability depends on choice of currency (the so-called *numéraire*).
- Before the 2016 US presidential election, investors predicted (correctly) that the value of the Mexican peso (in US dollars) would be substantially lower if Trump won than if Clinton won.

Extensions of risk neutral probability

- Definition of risk neutral probability depends on choice of currency (the so-called *numéraire*).
- Before the 2016 US presidential election, investors predicted (correctly) that the value of the Mexican peso (in US dollars) would be substantially lower if Trump won than if Clinton won.
- Given this, would the risk neutral probability of a Trump win have been higher with pesos as the numéraire or with dollars as the numéraire?

Extensions of risk neutral probability

- Definition of risk neutral probability depends on choice of currency (the so-called *numéraire*).
- Before the 2016 US presidential election, investors predicted (correctly) that the value of the Mexican peso (in US dollars) would be substantially lower if Trump won than if Clinton won.
- Given this, would the risk neutral probability of a Trump win have been higher with pesos as the numéraire or with dollars as the numéraire?
- Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define $P_{RN}(A)$ to be price of contract paying this amount if and when A occurs.

Extensions of risk neutral probability

- Definition of risk neutral probability depends on choice of currency (the so-called *numéraire*).
- Before the 2016 US presidential election, investors predicted (correctly) that the value of the Mexican peso (in US dollars) would be substantially lower if Trump won than if Clinton won.
- Given this, would the risk neutral probability of a Trump win have been higher with pesos as the numéraire or with dollars as the numéraire?
- Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define $P_{RN}(A)$ to be price of contract paying this amount if and when A occurs.
- For simplicity, we focus on fixed time T, fixed interest rate r in this lecture.

Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.

Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.

Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.
- Unlike “true probability” (what does that mean?) the “risk neutral probability” is an objectively measurable price.

Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.
- Unlike "true probability" (what does that mean?) the "risk neutral probability" is an objectively measurable price.
- Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.

Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.
- Unlike “true probability” (what does that mean?) the “risk neutral probability” is an objectively measurable price.
- Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.
- Listener: Then why not make some bets and get rich? If your estimates are so much better, law of large numbers says you’ll surely come out way ahead eventually.

Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.
- Unlike "true probability" (what does that mean?) the "risk neutral probability" is an objectively measurable price.
- Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.
- Listener: Then why not make some bets and get rich? If your estimates are so much better, law of large numbers says you'll surely come out way ahead eventually.
- Pundit: Well, you know... been busy... scruples about gambling... more to life than money...

Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.
- Unlike "true probability" (what does that mean?) the "risk neutral probability" is an objectively measurable price.
- Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.
- Listener: Then why not make some bets and get rich? If your estimates are so much better, law of large numbers says you'll surely come out way ahead eventually.
- Pundit: Well, you know... been busy... scruples about gambling... more to life than money...
- Listener: Yeah, that's what I thought.

Prices as expectations

- If r is risk free interest rate, then by definition, price of a contract paying dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$.

Prices as expectations

- If r is risk free interest rate, then by definition, price of a contract paying dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$.
- If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?

Prices as expectations

- If r is risk free interest rate, then by definition, price of a contract paying dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$.
- If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?
- Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.

Prices as expectations

- If r is risk free interest rate, then by definition, price of a contract paying dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$.
- If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?
- Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.
- Generally, in absence of arbitrage, price of contract that pays X at time T should be $E_{RN}(X)e^{-rT}$ where E_{RN} denotes expectation with respect to the risk neutral probability.

Prices as expectations

- If r is risk free interest rate, then by definition, price of a contract paying dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$.
- If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?
- Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.
- Generally, in absence of arbitrage, price of contract that pays X at time T should be $E_{RN}(X)e^{-rT}$ where E_{RN} denotes expectation with respect to the risk neutral probability.
- Example: if a non-divided paying stock will be worth X at time T, then its price today should be $E_{RN}(X)e^{-rT}$.

Prices as expectations

- If r is risk free interest rate, then by definition, price of a contract paying dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$.
- If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?
- Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.
- Generally, in absence of arbitrage, price of contract that pays X at time T should be $E_{RN}(X)e^{-rT}$ where E_{RN} denotes expectation with respect to the risk neutral probability.
- Example: if a non-divided paying stock will be worth X at time T, then its price today should be $E_{RN}(X)e^{-rT}$.
- In particular, the risk neutral expectation of tomorrow's (interest discounted) stock price is today's stock price.

Prices as expectations

- If r is risk free interest rate, then by definition, price of a contract paying dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$.
- If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?
- Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.
- Generally, in absence of arbitrage, price of contract that pays X at time T should be $E_{RN}(X)e^{-rT}$ where E_{RN} denotes expectation with respect to the risk neutral probability.
- Example: if a non-divided paying stock will be worth X at time T, then its price today should be $E_{RN}(X)e^{-rT}$.
- In particular, the risk neutral expectation of tomorrow's (interest discounted) stock price is today's stock price.
- Implies **fundamental theorem of asset pricing**, which says discounted price $\frac{X(n)}{A(n)}$ (where A is a risk-free asset) is a martingale with respected to **risk neutral probability**.

Outline

Outline

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \bigl(1 - F(x)\bigr) dx = E[X]$.

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \big(1 - F(x)\big)\, dx = E[X]$.
- So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \big(1 - F(x)\big)\,dx = E[X]$.
- So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.
- What is the meaning of $C(K) := \int_K^\infty \big(1 - F(x)\big)\,dx$?

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \big(1 - F(x)\big) dx = E[X]$.
- So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.
- What is the meaning of $C(K) := \int_K^\infty \big(1 - F(x)\big) dx$?
- It is area bounded between $y = F(x)$ and $y = 1$ and $x = K$.

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \big(1 - F(x)\big) dx = E[X]$.
- So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.
- What is the meaning of $C(K) := \int_K^\infty \big(1 - F(x)\big) dx$?
- It is area bounded between $y = F(x)$ and $y = 1$ and $x = K$.
- By translation argument, it is also $E[\max(X - K, 0)]$.

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \big(1 - F(x)\big) dx = E[X]$.
- So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.
- What is the meaning of $C(K) := \int_K^\infty \big(1 - F(x)\big) dx$?
- It is area bounded between $y = F(x)$ and $y = 1$ and $x = K$.
- By translation argument, it is also $E[\max(X - K, 0)]$.
- Note: $C'(x) = -\big(1 - F(x)\big) = F(x) - 1$ and $C''(x) = f(x)$.

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \big(1 - F(x)\big)dx = E[X]$.
- So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.
- What is the meaning of $C(K) := \int_K^\infty \big(1 - F(x)\big)dx$?
- It is area bounded between $y = F(x)$ and $y = 1$ and $x = K$.
- By translation argument, it is also $E[\max(X - K, 0)]$.
- Note: $C'(x) = -\big(1 - F(x)\big) = F(x) - 1$ and $C''(x) = f(x)$.
- Let's give C a name: we'll call it the **call function** of X.
  1. $C(K)$ is an expectation: $E[\max(X - K, 0)]$.
  2. $C(K)$ is area between $y = F(x)$ and $y = 1$ and $x = K$.
  3. $C(K)$ is an anti-anti-derivative of the density function f.

  Note that $C(0) = E[X]$ and $\lim_{K\to\infty} C(K) = 0$. C is convex with slope increasing from -1 to 0.

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \big(1 - F(x)\big)\,dx = E[X]$.
- So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.
- What is the meaning of $C(K) := \int_K^\infty \big(1 - F(x)\big)\,dx$?
- It is area bounded between $y = F(x)$ and $y = 1$ and $x = K$.
- By translation argument, it is also $E[\max(X - K, 0)]$.
- Note: $C'(x) = -\big(1 - F(x)\big) = F(x) - 1$ and $C''(x) = f(x)$.
- Let's give C a name: we'll call it the **call function** of X.
  1. $C(K)$ is an expectation: $E[\max(X - K, 0)]$.
  2. $C(K)$ is area between $y = F(x)$ and $y = 1$ and $x = K$.
  3. $C(K)$ is an anti-anti-derivative of the density function f.

  Note that $C(0) = E[X]$ and $\lim_{K\to\infty} C(K) = 0$. C is convex with slope increasing from -1 to 0.
- So now any random variable X comes with a pdf $f = f_X$, a cdf $F = F_X$ (an anti-derivative of f_X) and this call function $C = C_X$ (an anti-anti-derivative of f).

Call function: pretty cool whether you love finance or not

- **Recall:** if X is non-negative random variable with cumulative distribution function F, then $\int_0^\infty \big(1 - F(x)\big)\,dx = E[X]$.
- So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.
- What is the meaning of $C(K) := \int_K^\infty \big(1 - F(x)\big)\,dx$?
- It is area bounded between $y = F(x)$ and $y = 1$ and $x = K$.
- By translation argument, it is also $E[\max(X - K, 0)]$.
- Note: $C'(x) = -\big(1 - F(x)\big) = F(x) - 1$ and $C''(x) = f(x)$.
- Let's give C a name: we'll call it the **call function** of X.
  1. $C(K)$ is an expectation: $E[\max(X - K, 0)]$.
  2. $C(K)$ is area between $y = F(x)$ and $y = 1$ and $x = K$.
  3. $C(K)$ is an anti-anti-derivative of the density function f.

  Note that $C(0) = E[X]$ and $\lim_{K\to\infty} C(K) = 0$. C is convex with slope increasing from -1 to 0.
- So now any random variable X comes with a pdf $f = f_X$, a cdf $F = F_X$ (an anti-derivative of f_X) and this call function $C = C_X$ (an anti-anti-derivative of f).
- Wonder if C is good for anything....

Goals for today

- **Define:** $C(K) := \int_K^\infty (1 - F(x))dx = E[\max(X - K, 0)]$

Goals for today

- **Define:** $C(K) := \int_K^\infty (1 - F(x))dx = E[\max(X - K, 0)]$
- **Math goal:** understand C and how to compute it the special case that $X = e^N$, where N is a normal random variable.

Goals for today

- **Define:** $C(K) := \int_K^\infty (1 - F(x))dx = E[\max(X - K, 0)]$
- **Math goal:** understand C and how to compute it the special case that $X = e^N$, where N is a normal random variable.
- **Story goal:** give some financial motivation for all of this. Explain what C has to do with *option pricing* and what the special case $X = e^N$ has to do with the *Black-Scholes formula*.

Goals for today

- **Define:** $C(K) := \int_K^\infty (1 - F(x))dx = E[\max(X - K, 0)]$
- **Math goal:** understand C and how to compute it the special case that $X = e^N$, where N is a normal random variable.
- **Story goal:** give some financial motivation for all of this. Explain what C has to do with *option pricing* and what the special case $X = e^N$ has to do with the *Black-Scholes formula*.
- **Weird fact:** If X is a real world random quantity (such as the price of gold or euros or stock shares at a future date) and we use risk neutral probability, then sometimes the call function C (or a related "put function") is what we can look up online. One then uses the quoted C values to work out F_X and f_X.

Goals for today

- **Define:** $C(K) := \int_K^\infty (1 - F(x))dx = E[\max(X - K, 0)]$
- **Math goal:** understand C and how to compute it the special case that $X = e^N$, where N is a normal random variable.
- **Story goal:** give some financial motivation for all of this. Explain what C has to do with *option pricing* and what the special case $X = e^N$ has to do with the *Black-Scholes formula*.
- **Weird fact:** If X is a real world random quantity (such as the price of gold or euros or stock shares at a future date) and we use risk neutral probability, then sometimes the call function C (or a related "put function") is what we can look up online. One then uses the quoted C values to work out F_X and f_X.
- **Grand story goal:** Say something about the link between probability and the real world. What is the probability that price of Microsoft stock will rise by more than ten dollars over the next month? What is the probability that price of oil will drop more than ten percent next year? How can I (using internet and math) come up with a reasonable answer?

European call options

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

> The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

European call options

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

- If X is time T stock price, then value of option at time T is $g(X) = \max\{0, X - K\}$. If we use the risk neutral probability measure, then the price now should be

$$e^{-rT}E[g(X)] = e^{-rT}C(K),$$

where C is the call function corresponding to X.

European call options

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

  The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

- If X is time T stock price, then value of option at time T is $g(X) = \max\{0, X - K\}$. If we use the risk neutral probability measure, then the price now should be

$$e^{-rT}E[g(X)] = e^{-rT}C(K),$$

  where C is the call function corresponding to X.
- Recall first-slide observation:

$$C'(K) = F_X(K) - 1 \quad , \quad C''(K) = f_X(K).$$

European call options

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

- If X is time T stock price, then value of option at time T is $g(X) = \max\{0, X - K\}$. If we use the risk neutral probability measure, then the price now should be

$$e^{-rT}E[g(X)]) = e^{-rT}C(K),$$

where C is the call function corresponding to X.
- Recall first-slide observation:

$$C'(K) = F_X(K) - 1 \quad , \quad C''(K) = f_X(K).$$

- Can look up $C(K)$ values for stock (say GOOG) at cboe.com, apply smoothing, take derivatives, approximate F_X and f_X.

European put options

- **European put option** gives holder write to *sell* stock for K dollars at time T.

European put options

- **European put option** gives holder write to *sell* stock for K dollars at time T.
- Analysis is basically the same as for call options except that one replaces the "call function" $C(K) = E[\max(X - K, 0)]$ with the "put function" defined by

$$P(K) = E[\max(K - X, 0)].$$

European put options

- **European put option** gives holder write to *sell* stock for K dollars at time T.
- Analysis is basically the same as for call options except that one replaces the "call function" $C(K) = E[\max(X-K,0)]$ with the "put function" defined by

$$P(K) = E[\max(K-X,0)].$$

- $\max(a,0) - \max(-a,0) = a$. So $C(K) - P(K) = E[X-K]$.

$$P(K) = C(K) - E[X] + K = \int_0^K F(x)dx.$$

European put options

- **European put option** gives holder write to *sell* stock for K dollars at time T.
- Analysis is basically the same as for call options except that one replaces the "call function" $C(K) = E[\max(X - K, 0)]$ with the "put function" defined by

$$P(K) = E[\max(K - X, 0)].$$

- $\max(a, 0) - \max(-a, 0) = a$. So $C(K) - P(K) = E[X - K]$.

$$P(K) = C(K) - E[X] + K = \int_0^K F(x)dx.$$

- The put function is an anti-anti-derivative of f (like the call function) but it has a slope that increases from 0 to 1 (instead of from -1 to 0) and it satisfies $P(0) = 0$.

European put options

- **European put option** gives holder write to *sell* stock for K dollars at time T.
- Analysis is basically the same as for call options except that one replaces the "call function" $C(K) = E[\max(X-K,0)]$ with the "put function" defined by

$$P(K) = E[\max(K-X,0)].$$

- $\max(a,0) - \max(-a,0) = a$. So $C(K) - P(K) = E[X-K]$.

$$P(K) = C(K) - E[X] + K = \int_0^K F(x)dx.$$

- The put function is an anti-anti-derivative of f (like the call function) but it has a slope that increases from 0 to 1 (instead of from -1 to 0) and it satisfies $P(0) = 0$.
- Many trading platforms sell call and put options side by side.

European put options

- **European put option** gives holder write to *sell* stock for K dollars at time T.
- Analysis is basically the same as for call options except that one replaces the "call function" $C(K) = E[\max(X - K, 0)]$ with the "put function" defined by

$$P(K) = E[\max(K - X, 0)].$$

- $\max(a, 0) - \max(-a, 0) = a$. So $C(K) - P(K) = E[X - K]$.

$$P(K) = C(K) - E[X] + K = \int_0^K F(x)dx.$$

- The put function is an anti-anti-derivative of f (like the call function) but it has a slope that increases from 0 to 1 (instead of from -1 to 0) and it satisfies $P(0) = 0$.
- Many trading platforms sell call and put options side by side.
- For simplicity we focus on call functions in this lecture.

Outline

Martingales and stopping times

Risk neutral probability and martingales

Call function

Black-Scholes

Outline

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
- 1997 Nobel Prize.

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
- 1997 Nobel Prize.
- **Assumption:** the log of an asset price X at fixed future time T is a normal random variable (call it N) with some known variance (call it $T\sigma^2$) and some mean (call it μ) with respect to risk neutral probability.

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
- 1997 Nobel Prize.
- **Assumption:** the log of an asset price X at fixed future time T is a normal random variable (call it N) with some known variance (call it $T\sigma^2$) and some mean (call it μ) with respect to risk neutral probability.
- **Observation:** N normal $(\mu, T\sigma^2)$ implies $E[e^N] = e^{\mu + T\sigma^2/2}$.

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
- 1997 Nobel Prize.
- **Assumption:** the log of an asset price X at fixed future time T is a normal random variable (call it N) with some known variance (call it $T\sigma^2$) and some mean (call it μ) with respect to risk neutral probability.
- **Observation:** N normal $(\mu, T\sigma^2)$ implies $E[e^N] = e^{\mu + T\sigma^2/2}$.
- **Observation:** If X_0 is the current price then $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$.

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
- 1997 Nobel Prize.
- **Assumption:** the log of an asset price X at fixed future time T is a normal random variable (call it N) with some known variance (call it $T\sigma^2$) and some mean (call it μ) with respect to risk neutral probability.
- **Observation:** N normal $(\mu, T\sigma^2)$ implies $E[e^N] = e^{\mu + T\sigma^2/2}$.
- **Observation:** If X_0 is the current price then $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$.
- **Observation:** This implies $\mu = \log X_0 + (r - \sigma^2/2)T$.

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
- 1997 Nobel Prize.
- **Assumption:** the log of an asset price X at fixed future time T is a normal random variable (call it N) with some known variance (call it $T\sigma^2$) and some mean (call it μ) with respect to risk neutral probability.
- **Observation:** N normal $(\mu, T\sigma^2)$ implies $E[e^N] = e^{\mu + T\sigma^2/2}$.
- **Observation:** If X_0 is the current price then $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu+(\sigma^2/2-r)T}$.
- **Observation:** This implies $\mu = \log X_0 + (r - \sigma^2/2)T$.
- **General Black-Scholes conclusion:** If g is any function then the price of a contract that pays $g(X)$ at time T is

$$E[g(e^N)]e^{-rT}$$

where N is normal with mean μ and variance $T\sigma^2$.

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
- 1997 Nobel Prize.
- **Assumption:** the log of an asset price X at fixed future time T is a normal random variable (call it N) with some known variance (call it $T\sigma^2$) and some mean (call it μ) with respect to risk neutral probability.
- **Observation:** N normal $(\mu, T\sigma^2)$ implies $E[e^N] = e^{\mu + T\sigma^2/2}$.
- **Observation:** If X_0 is the current price then $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$.
- **Observation:** This implies $\mu = \log X_0 + (r - \sigma^2/2)T$.
- **General Black-Scholes conclusion:** If g is any function then the price of a contract that pays $g(X)$ at time T is

$$E[g(e^N)]e^{-rT}$$

  where N is normal with mean μ and variance $T\sigma^2$.
- **Surprise:** No need to guess μ. It is fixed by X_0, r, σ, T.

Black-Scholes for European call option

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

Black-Scholes for European call option

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

- **Recall:** If X is time T stock price, then value of option at time T is $g(X) = \max\{0, X - K\}$. Price now should be

$$e^{-rT} E_{RN} g(X) = e^{-rT} C(K).$$

Black-Scholes for European call option

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

- **Recall:** If X is time T stock price, then value of option at time T is $g(X) = \max\{0, X - K\}$. Price now should be

$$e^{-rT} E_{RN} g(X) = e^{-rT} C(K).$$

- **Black-Scholes:** this is $e^{-rT} E[g(e^N)]$ where N is normal with variance $T\sigma^2$ and mean $\mu = \log X_0 + (r - \sigma^2/2)T$.

Black-Scholes for European call option

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

- **Recall:** If X is time T stock price, then value of option at time T is $g(X) = \max\{0, X - K\}$. Price now should be

$$e^{-rT} E_{RN} g(X) = e^{-rT} C(K).$$

- **Black-Scholes:** this is $e^{-rT} E[g(e^N)]$ where N is normal with variance $T\sigma^2$ and mean $\mu = \log X_0 + (r - \sigma^2/2)T$.
- Write this as

$$e^{-rT} E[\max\{0, e^N - K\}] = e^{-rT} E[(e^N - K)1_{N \geq \log K}]$$

$$= \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) dx.$$

The famous formula

- Let T be time to maturity, X_0 current price of underlying asset, K strike price, r risk free interest rate, σ the volatility.

The famous formula

- Let T be time to maturity, X_0 current price of underlying asset, K strike price, r risk free interest rate, σ the volatility.
- We need to compute $\frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}}(e^x - K)dx$ where $\mu = rT + \log X_0 - T\sigma^2/2$.

The famous formula

- Let T be time to maturity, X_0 current price of underlying asset, K strike price, r risk free interest rate, σ the volatility.
- We need to compute $\frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}}(e^x - K)dx$ where $\mu = rT + \log X_0 - T\sigma^2/2$.
- Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function Φ.

The famous formula

- Let T be time to maturity, X_0 current price of underlying asset, K strike price, r risk free interest rate, σ the volatility.
- We need to compute $\frac{e^{-rT}}{\sigma\sqrt{2\pi T}}\int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}}(e^x - K)dx$ where $\mu = rT + \log X_0 - T\sigma^2/2$.
- Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function Φ.
- Price of European call is $\Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT}$ where $d_1 = \frac{\ln(\frac{X_0}{K}) + (r + \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln(\frac{X_0}{K}) + (r - \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$.

Perspective: implied volatility

- Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.

Perspective: implied volatility

- Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.
- "Implied volatility" is the value of σ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.

Perspective: implied volatility

- Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.
- "Implied volatility" is the value of σ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.
- If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices K. In practice, when the implied volatility is viewed as a function of K (sometimes called the "volatility smile"), it is not constant.

Perspective: implied volatility

- Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.
- "Implied volatility" is the value of σ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.
- If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices K. In practice, when the implied volatility is viewed as a function of K (sometimes called the "volatility smile"), it is not constant.
- Nonetheless, "implied volatility" has become a standard part of the finance lexicon. When traders want to get a rough sense of how a financial derivative is priced, they often ask for the implied volatility (a number automatically computed in many financial software packages).

Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.

Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.
- **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large T.

Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.
- **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large T.
- **Replicating portfolio point of view:** in simple models (e.g., where wealth always goes up or down by fixed factor each day) can transfer money between the stock and the risk free asset to ensure our wealth at time T equals option payout. Option price is required initial investment, which is risk neutral expectation of payout.

Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.
- **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large T.
- **Replicating portfolio point of view:** in simple models (e.g., where wealth always goes up or down by fixed factor each day) can transfer money between the stock and the risk free asset to ensure our wealth at time T equals option payout. Option price is required initial investment, which is risk neutral expectation of payout.
- **Where arguments for assumption break down:** Fluctuation sizes vary from day to day. Prices can have big jumps. Past volatility does not determine future volatility.
- **Fixes:** variable volatility, random interest rates, Lévy jumps....