18.600: Lecture 35
Martingales and risk neutral probability

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Outline

Martingales and stopping times

Risk neutral probability and martingales

Call function

Black-Scholes
Martingales and stopping times

Risk neutral probability and martingales

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Black-Scholes
Let $S$ be the probability space. Let $X_0, X_1, X_2, \ldots$ be a sequence of real random variables. Interpret $X_i$ as price of asset at $i$th time step.
Recall martingale definition

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"Given all I know today, expected price tomorrow is the price today."
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- “Given all I know today, expected price tomorrow is the price today.”
Recall stopping time definition

- Let $T$ be a non-negative integer valued random variable.

- Think of $T$ as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.

- Say that $T$ is a stopping time if the event that $T = n$ depends only on the values $X_i$ for $i \leq n$. In other words, the decision to sell at time $n$ depends only on prices up to time $n$, not on (as yet unknown) future prices.

- Optional stopping theorem: As long as $X_T$ is bounded (or $T$ is bounded) between fixed constants with probability one, we have $E[X_T] = E[X_0]$.

- Informal proof if $P(T \leq N) = 1$ for some $N$: OST says “If only care about expectation, selling at time 0 is as good as any strategy for selling at a time between 0 and $N$.” If we make it to time $N-1$ then at that point we may as well sell, since martingale property implies $E[X_{N-1}|F_{N-1}] = X_{N-1}$.

- Replace $N$ by $N-1$, proceed inductively.
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According to the **fundamental theorem of asset pricing**, the discounted price $\frac{X(n)}{A(n)}$, where $A$ is a risk-free asset, is a martingale with respect to **risk neutral probability**.
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For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?

If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.
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Assume bid-ask spread essentially zero. (Otherwise risk neutral probability would be somewhere between bid and ask, wouldn’t know where.)
Risk neutral probability of outcomes known at fixed time \( T \)

- **Risk neutral probability of event** \( A \): \( P_{RN}(A) \) denotes

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\frac{\text{Price\{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs \}}} {\text{Price\{Contract paying 1 dollar at time } T \text{ no matter what \}}}.\]

- If risk-free interest rate is constant and equal to \( r \) (compounded continuously), then denominator is \( e^{-rT} \).

- Assuming no arbitrage (i.e., no risk-free profit with zero upfront investment), \( P_{RN} \) satisfies axioms of probability. That is, \( 0 \leq P_{RN}(A) \leq 1 \), and \( P_{RN}(S) = 1 \), and if events \( A_j \) are disjoint then

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P_{RN}(A_1 \cup A_2 \cup ... ) = P_{RN}(A_1) + P_{RN}(A_2) + ... .\]

- Arbitrage example: if \( A \) and \( B \) are disjoint and \( P_{RN}(A \cup B) < P(A) + P(B) \) then we sell contracts paying 1 if \( A \) occurs and 1 if \( B \) occurs, buy contract paying 1 if \( A \cup B \) occurs, pocket difference.
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At first sight, one might think that $P_{RN}(A)$ describes the market’s best guess at the probability that $A$ will occur.

But suppose $A$ is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$? Should be 0. Even if people think $A$ is likely, a contract paying a dollar when $A$ occurs is worthless.

Now, suppose there are only 2 outcomes: $A$ is event that economy booms and everyone prospers and $B$ is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think $A$ has a $\frac{1}{2}$ chance to occur, do we expect $P_{RN}(A) > \frac{1}{2}$ or $P_{RN}(A) < \frac{1}{2}$?

Answer: $P_{RN}(A) < \frac{1}{2}$. People are risk averse. In second scenario they need the money more.
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Answer: $P_{RN}(A) < .5$. People are risk averse. In second scenario they need the money more.
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Arguably yes. The amount that *people in general* need or value dollars does not depend much on whether $A$ occurs (even though the financial needs of specific individuals may depend on heavily on $A$).
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Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to team’s prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrages to keep price near a reasonable guess of what well-informed informed experts would consider the true probability.
Extensions of risk neutral probability

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Given this, would the risk neutral probability of a Trump win have been higher with pesos as the numéraire or with dollars as the numéraire?
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Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define \( P_{RN}(A) \) to be price of contract paying this amount if and when \( A \) occurs.
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For simplicity, we focus on fixed time $T$, fixed interest rate $r$ in this lecture.
Risk neutral probability is objective

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- Listener: Yeah, that’s what I thought.
Prices as expectations

If $r$ is risk free interest rate, then by definition, price of a contract paying dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$. 

If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?

Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.

Generally, in absence of arbitrage, price of contract that pays $X$ at time $T$ should be $E_{RN}(X)e^{-rT}$ where $E_{RN}$ denotes expectation with respect to the risk neutral probability.

Example: if a non-divided paying stock will be worth $X$ at time $T$, then its price today should be $E_{RN}(X)e^{-rT}$.

In particular, the risk neutral expectation of tomorrow's (interest discounted) stock price is today's stock price. 

Implies fundamental theorem of asset pricing, which says discounted price $X^n A^n$ (where $A$ is a risk-free asset) is a martingale with respected to risk neutral probability.
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- In particular, the risk neutral expectation of tomorrow’s (interest discounted) stock price is today’s stock price.
- Implies fundamental theorem of asset pricing, which says discounted price $\frac{X(n)}{A(n)}$ (where $A$ is a risk-free asset) is a martingale with respected to risk neutral probability.
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Call function: pretty cool whether you love finance or not

- **Recall:** if $X$ is a non-negative random variable with cumulative distribution function $F$, then $\int_0^\infty (1 - F(x)) dx = E[X]$. 

- Let's give $C$ a name: we'll call it the call function of $X$. 
  1. $C(K)$ is an expectation: $E[\max(X - K, 0)]$. 
  2. $C(K)$ is area between $y = F(x)$ and $y = 1$ and $x = K$. 
  3. $C(K)$ is an anti-anti-derivative of the density function $f$. 

- Note that $C(0) = E[X]$ and $\lim_{K \to \infty} C(K) = 0$. $C$ is convex with slope increasing from $-1$ to $0$. 

- So now any random variable $X$ comes with a pdf $f_X = f$, a cdf $F_X = F$ (an anti-derivative of $f_X$) and this call function $C = C_X$ (an anti-anti-derivative of $f_X$). 

- Wonder if $C$ is good for anything....
Call function: pretty cool whether you love finance or not

- **Recall:** if $X$ is non-negative random variable with cumulative distribution function $F$, then $\int_0^\infty (1 - F(x)) \, dx = E[X]$.
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Wonder if $C$ is good for anything....
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So $E[X]$ is area between $y = F(x)$ and $y = 1$ and $x = 0$.

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- **Weird fact:** If \( X \) is a real world random quantity (such as the price of gold or euros or stock shares at a future date) and we use risk neutral probability, then sometimes the call function \( C \) (or a related “put function”) is what we can look up online. One then uses the quoted \( C \) values to work out \( F_X \) and \( f_X \).

- **Grand story goal:** Say something about the link between probability and the real world. What is the probability that price of Microsoft stock will rise by more than ten dollars over the next month? What is the probability that price of oil will drop more than ten percent next year? How can I (using internet and math) come up with a reasonable answer?
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If \( X \) is time \( T \) stock price, then value of option at time \( T \) is \( g(X) = \max\{0, X - K\} \). If we use the risk neutral probability measure, then the price now should be \( e^{-rT}E[g(X)] = e^{-rT}C(K) \), where \( C \) is the call function corresponding to \( X \).

Recall first-slide observation:

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Analysis is basically the same as for call options except that one replaces the "call function" $C(K) = E[\max(X-K,0)]$ with the "put function" defined by

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The put function is an anti-anti-derivative of $f$ (like the call function) but it has a slope that increases from 0 to 1 (instead of from $-1$ to 0) and it satisfies $P(0) = 0$. Many trading platforms sell call and put options side by side. For simplicity we focus on call functions in this lecture.
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Outline

Martingales and stopping times

Risk neutral probability and martingales

Call function

Black-Scholes
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Black-Scholes
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1997 Nobel Prize.

Assumption:

\[ \text{the log of an asset price } X \text{ at fixed future time } T \text{ is a normal random variable (call it } N \text{) with some known variance (call it } T \sigma^2 \text{) and some mean (call it } \mu \text{) with respect to risk neutral probability.} \]

Observation:

\[ N \text{ normal } (\mu, T \sigma^2) \text{ implies } E[e^N] = e^{\mu + \frac{T \sigma^2}{2}}. \]

Observation:

If \( X_0 \) is the current price then

\[ X_0 = E[R e^N] = E[R e^{\mu + \frac{T \sigma^2}{2} - r T}] = e^{\mu + \left( \frac{\sigma^2}{2} - r \right) T}. \]

Observation:

This implies \( \mu = \log X_0 + \left( r - \frac{\sigma^2}{2} \right) T \).

General Black-Scholes conclusion:

If \( g \) is any function then the price of a contract that pays \( g(X) \) at time \( T \) is

\[ E[g(e^N)] e^{-r T} \text{ where } N \text{ is normal with mean } \mu \text{ and variance } T \sigma^2. \]

Surprise:

No need to guess \( \mu \). It is fixed by \( X_0, r, \sigma, T \).
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Write this as

$$e^{-rT} E[\max\{0, e^N - K\}] = e^{-rT} E[(e^N - K)1_{N \geq \log K}]$$

$$= \frac{e^{-rT}}{\sigma \sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) dx.$$
Let $T$ be time to maturity, $X_0$ current price of underlying asset, $K$ strike price, $r$ risk free interest rate, $\sigma$ the volatility.
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where

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\mu = rT + \log X_0 - T \sigma^2 / 2.
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The famous formula

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- Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function $\Phi$. 
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\frac{e^{-rT}}{\sigma \sqrt{2 \pi T}} \int_{\log K}^{\infty} e^{-(x-\mu)^2/(2T\sigma^2)} (e^x - K) \, dx
\]

where \( \mu = rT + \log X_0 - T\sigma^2/2 \).

Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function \( \Phi \).

Price of European call is \( \Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT} \) where

\[
d_1 = \frac{\ln(\frac{X_0}{K})+(r+\frac{\sigma^2}{2})(T)}{\sigma \sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(\frac{X_0}{K})+(r-\frac{\sigma^2}{2})(T)}{\sigma \sqrt{T}}.
\]
Perspective: implied volatility

- Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.
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- Nonetheless, “implied volatility” has become a standard part of the finance lexicon. When traders want to get a rough sense of how a financial derivative is priced, they often ask for the implied volatility (a number automatically computed in many financial software packages).
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Perspective: why is Black-Scholes not exactly right?

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- **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large $T$. 

- **Fixes:** variable volatility, random interest rates, Lévy jumps....
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Replicating portfolio point of view: in simple models (e.g., where wealth always goes up or down by fixed factor each day) can transfer money between the stock and the risk free asset to ensure our wealth at time \( T \) equals option payout. Option price is required initial investment, which is risk neutral expectation of payout.
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- **Where arguments for assumption break down:** Fluctuation sizes vary from day to day. Prices can have big jumps. Past volatility does not determine future volatility.

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