Permutations, Poker, and Powerball

18.600 Problem Set 1, due September 13

Welcome to your first 18.600 problem set! There will be ten problem sets this semester, each with a different theme, and each including a mix of problems of my own design and problems from the Sheldon Ross 8th edition textbook. Before we begin the problems, let me provide some basic information and experience-based advice to help you get more out of the course as a whole.

1. Recognize that, like courses in the 18.0x series, this is a fundamental course, meant to be accessible to students from every department at MIT. On the other hand (and here it is crucial to set expectations correctly) the course is *very challenging* for many students, and in some ways more advanced than the 18.0x courses. You should not be surprised or disappointed when you don't know how to do the problems right away, or when you need to consult a second or third source to understand a concept.

2. Attend lectures! In principle, it may be possible to learn much of the course material by clicking through lecture slides in Adobe fullscreen mode on your bedroom laptop, but this will not necessarily make you happy. People who come to lecture and stay engaged (ask questions, answer questions, etc.) learn more quickly, have more fun, and remember longer. If the lectures seem a little fast, try reading the textbook and/or slides in advance and come prepared to ask questions. If they seem a little slow (or if the topic is one you have seen before) try just showing up and letting the lecture be your first exposure. The fraction of students who have seen the lecture material before will be significant during the first few weeks, but will decline very quickly after that.

3. Use the textbook. Part of the reason for including problems from the textbook is to remind you that the textbook exists, and to encourage you to read it (any of the 6th through 10th editions will do). All of the course material (except for the parts about martingales and Black-Scholes, which will be covered in a separate handout) is covered in the textbook. There are many inexpensive ways to get electronic or hard copies of the textbook online (check out ebay, amazon, google, etc.)

4. Start the problem sets early and come to office hours if you have questions. The problem sets are more challenging than the exams and they serve a different purpose. They are meant to be educational in their own right. If you are one of the lucky few for whom the problems are easy, you will still learn a lot by thinking through the concepts and applications. You are free to collaborate with other students, look up material on the internet or in books, offer each other hints (though

not full solutions or answers) on Piazza, and ask me and the TAs for ideas during office hours and recitations. Note however that some of the problems are reused from prior years, and you are definitely *not* allowed to access or consult prior year problem set solutions. (Prior year *exam* solutions, on the other hand, are posted on the public course webpage, and you are welcome to use those.)

5. If you are doing well in the course, try to help out by answering (as well as asking) questions on Piazza. This will help solidify your own understanding and will be appreciated by fellow students (as well as your TAs and me). I am going to try to restrain myself from answering most basic math questions on Piazza (so students have more of a chance to answer questions for each other) but if you post a question on Piazza and it remains unanswered after 48 hours, let me know by email and I'll look into it. If you have a personal problem or a complaint or a request (e.g., "Could you use a different chalk color?"), you should email the TAs or me directly, rather than posting something on Piazza, which will be more of a public forum.

6. Spare at least a *little* time for thinking and exploring that has nothing to do with your grade. Like looking up and reading a bit more about mathematical or practical issues raised in problem sets. Ponder some big questions about applications. What are we doing wrong in medicine? In traffic management? In college admissions? In teaching? In food preparation? Is there simple advice that, if followed, would make us all better off? Is there other commonly accepted advice that we should all stop following? How can we find out what these things are? How can probability help? As the course progresses, we will see many problems with applications; but each problem is the beginning of a conversation, not the end.

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Now the problems. The first problem set is about basic combinatorics. Cards, hats, permutations, balls, binomial and multinomial coefficients. It will help to keep these stories in mind as the course progresses.

Please stop by my weekly office hours (2-249, Wednesday 3 to 5) for discussion. I have a large office with room for several people to work and talk in a relaxed way. I'm also usually free just after lecture on Monday and Friday if you want to stop me to ask a question outside of 10-250.

A. FROM ROSS 8th EDITION CHAPTER ONE:

1. **Problem 10:** In how many ways can 8 people be seated in a row if

(a) there are no restrictions on the seating arrangement?

(b) persons A and B must sit next to each other?

(c) there are 4 women and 4 men and no 2 men or 2 women can sit next to each other?

(d) there are 5 men and they must sit next to each other?

(e) there are 4 married couples and each couple must sit together?

2. **Theoretical Exercise 11:** The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \quad n \geq k.$$

Give a combinatorial argument (no computations are needed) to establish this identity. *Hint:* Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?

B. Consider permutations $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. There are $n!$ such permutations altogether. Of these permulations...

1. How many have only one cycle, i.e., have the property that $\sigma(1), \sigma \circ \sigma(1), \sigma \circ \sigma \circ \sigma(1), \ldots$ cycles through all elements of $\{1, 2, \ldots, n\}$?

2. How many have exactly two cycles, one of length k (where $1 \leq k \leq n-1$) and one of length $n-k$?

3. How many are involutions, i.e., have the property that for each j we have $\sigma \circ \sigma(j) = j$? (*Hint:* Argue that if σ is an involution then each j is either a fixed point — i.e., satisfies $\sigma(j) = j$ — or part of a cycle of length two. Compute the number of involutions with exactly k cycles of length 2, and then write your overall answer as a sum over k.)

C. In a standard deck of 52 cards, there are 4 suits, and 13 cards of each suit. Using such a deck, there are $\binom{52}{13}$ ways to form a bridge hand containing 13 cards. How many of these hands have the property that:

1. All 13 cards belong to the same suit.

2. Exactly 2 of the 4 suits are represented in the hand.

3. Exactly 3 of the 4 suits are represented in the hand.

4. All 4 of the suits are represented (i.e., there is at least one card of each suit).

There is a hint on the next page, but don't look before you need to.

Hint: This one is legitimately tricky. If it helps, you can try generalizing the problem. Imagine that instead of 4 suits you have a deck with k suits, 13 cards of each suit—and then let $N_k(m)$ be the number of ways to produce a 13 card hand *from this deck* that has exactly m suits represented. Maybe you can build a table containing all the values of $N_k(m)$ for $k \in \{1,2,3,4\}$ and $1 \le m \le k$. Can you show that $\sum_{m=1}^{k} N_k(m) = \binom{13k}{13}$? Can you show that $N_k(m) = \binom{k}{m} N_m(m)$? Does this help you complete the table?

D. In the US lottery game of Powerball one is required to choose an (unordered) collection of five numbers from the set $\{1,2,\dots,69\}$ (the white balls) along with another number from the set $\{1,2,\dots,26\}$ (the red ball). So there are $\binom{69}{5} \cdot 26 = 292201338$ possible Powerball outcomes. You make your selection (five white, one red), the Powerball people choose theirs randomly, and you win if there is a match. Suppose that you have already chosen your numbers (the unordered set of five white, and the one red). How many possible Powerball outcomes match *exactly one* of your five white numbers (regardless of whether they match the red number)? How many match exactly two of your five white numbers? How many match exactly three? How many match one red ball *plus* exactly two white balls? Now, divide each of these numbers by 292201338 to produce a *probability* of seeing that outcome and use a calculator to give a numerical value. Write a sentence about what seems interesting or surprising about these values.

Remark: People in the US spend over 70 billion per year on lottery tickets (about 32 percent of which is returned in big lottery payouts). That's over two hundred dollars per person, with many players spending thousands of dollars a year. The psychological appeal may be hard for us to understand. But the fact that regular players get "close" now and then (matching two or three numbers) may be part of what keeps them coming back. If you are one of the *many* people who buys more than 1000 lottery tickets per year, you will probably match four of the six balls at some point during your life. If you have a dozen friends who do the same thing, one of them will probably match five of six balls at some point, which will *seem* very close. You can double check your computations by looking up the odds on the Powerball wikipedia page.

E. Derive the following formulas, which will be useful later in this course:

1. **Normal density formula:** $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = 1$. (Multiply both sides by $\sqrt{2\pi}$ and square both sides to get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy = 2\pi.$$

Then derive this by computing the integral in polar coordinates. You can look up the derivation in the book if you get stuck.)

2. **Poisson mass formula:** $\sum_{k=0}^{\infty} e^{-\lambda}\lambda^k/k! = 1$ if $\lambda > 0$. Hint: recall (or look up) the Taylor expansion for the function $f(\lambda) = e^{\lambda}$.

3. **Binomial sum formula:** $\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = 1$ where $p + q = 1$ and n is a positive integer. Hint: try expanding $(p+q)^n$.

4. **Factorial formula:** $\int_0^{\infty} x^n e^{-x} dx = n!$. (Assume $n \geq 0$ is an integer and use integration by parts and induction.)

Store these formulas in long term memory and write "Got it!"

Remark: You will at some point have to learn a few formulas for this class: in particular, those that appear in red on the so-called story sheet posted on the public course webpage. But it will turn out that a surprising number of them (perhaps a majority) are obtained in some way from the four formulas listed above. Internalizing these few facts now will help you a lot going forward. These formulas (or close variants) are among those appearing in garageband clip about identities I posted last year at `http://math.mit.edu/~sheffield/2018600/kindofthing.mp4`. I am not sure about its musical merits, but the clip does contain some nice identities, as well as some problems that you will see later in this course.