

18.600: Lecture 15

Lectures 1-14 Review

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Outline

Counting tricks and basic principles of probability

Discrete random variables

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- ▶ Answer: $\binom{n+k-1}{n}$. Represent partition by $k-1$ bars and n stars, e.g., as $** \mid ** \mid \mid **** \mid *$.

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- ▶ Finite additivity: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.
- ▶ Countable additivity: $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \emptyset$ for each pair i and j .

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- ▶ More generally,

$$\begin{aligned} P(\cup_{i=1}^n E_i) &= \sum_{i=1}^n P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \dots \\ &\quad + (-1)^{(r+1)} \sum_{i_1 < i_2 < \dots < i_r} P(E_{i_1} E_{i_2} \dots E_{i_r}) \\ &= + \dots + (-1)^{n+1} P(E_1 E_2 \dots E_n). \end{aligned}$$

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- ▶ The notation $\sum_{i_1 < i_2 < \dots < i_r}$ means a sum over all of the $\binom{n}{r}$ subsets of size r of the set $\{1, 2, \dots, n\}$.

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- ▶ $P(\cup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots \pm \frac{1}{n!}$
- ▶ $1 - P(\cup_{i=1}^n E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots \pm \frac{1}{n!} \approx 1/e \approx .36788$

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- ▶ Nice fact: $P(E_1 E_2 E_3 \dots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \dots P(E_n|E_1 \dots E_{n-1})$

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- ▶ Useful when we think about multi-step experiments.
- ▶ For example, let E_i be event i th person gets own hat in the n -hat shuffle problem.

Dividing probability into two cases



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- ▶ In words: want to know the probability of E . There are two scenarios F and F^c . If I know the probabilities of the two scenarios and the probability of E conditioned on each scenario, I can work out the probability of E .

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- ▶ Tells how to update estimate of probability of A when new evidence restricts your sample space to B .
- ▶ So $P(A|B)$ is $\frac{P(B|A)}{P(B)}$ times $P(A)$.
- ▶ Ratio $\frac{P(B|A)}{P(B)}$ determines “how compelling new evidence is”.

$P(\cdot|F)$ is a probability measure

- ▶ We can check the probability axioms: $0 \leq P(E|F) \leq 1$, $P(S|F) = 1$, and $P(\cup E_i) = \sum P(E_i|F)$, if i ranges over a countable set and the E_i are disjoint.

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- ▶ The probability measure $P(\cdot|F)$ is related to $P(\cdot)$.
- ▶ To get former from latter, we set probabilities of elements outside of F to zero and multiply probabilities of events inside of F by $1/P(F)$.
- ▶ $P(\cdot)$ is the *prior* probability measure and $P(\cdot|F)$ is the *posterior* measure (revised after discovering that F occurs).

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- ▶ Equivalent statement: $P(E|F) = P(E)$. Also equivalent: $P(F|E) = P(F)$.

Independence of multiple events

- Say $E_1 \dots E_n$ are independent if for each $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$ we have $P(E_{i_1} E_{i_2} \dots E_{i_k}) = P(E_{i_1})P(E_{i_2}) \dots P(E_{i_k})$.

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- ▶ In other words, the product rule works.
- ▶ Independence implies $P(E_1 E_2 E_3 | E_4 E_5 E_6) = \frac{P(E_1) P(E_2) P(E_3) P(E_4) P(E_5) P(E_6)}{P(E_4) P(E_5) P(E_6)} = P(E_1 E_2 E_3)$, and other similar statements.

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- ▶ Does pairwise independence imply independence?
- ▶ No. Consider these three events: first coin heads, second coin heads, odd number heads. Pairwise independent, not independent.

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- ▶ For each a in this countable set, write $p(a) := P\{X = a\}$. Call p the **probability mass function**.
- ▶ Write $F(a) = P\{X \leq a\} = \sum_{x \leq a} p(x)$. Call F the **cumulative distribution function**.

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- ▶ Example: in n -hat shuffle problem, let E_i be the event i th person gets own hat.
- ▶ Then $\sum_{i=1}^n 1_{E_i}$ is total number of people who get own hats.

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- ▶ Represents weighted average of possible values X can take, each value being weighted by its probability.

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- ▶ Agrees with the **SUM OVER POSSIBLE X VALUES** definition:

$$E[X] = \sum_{x:p(x)>0} xp(x).$$

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- ▶ How can we compute $E[g(X)]$?
- ▶ Answer:

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x).$$

Additivity of expectation

- ▶ If X and Y are distinct random variables, then $E[X + Y] = E[X] + E[Y]$.

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- ▶ This is called the **linearity of expectation**.
- ▶ Can extend to more variables
$$E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n].$$

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- ▶ Variance is one way to measure the amount a random variable “varies” from its mean over successive trials.
- ▶ Very important alternate formula: $\text{Var}[X] = E[X^2] - (E[X])^2$.

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- ▶ Also, $\text{Var}[aX] = a^2 \text{Var}[X]$.
- ▶ Proof: $\text{Var}[aX] = E[a^2 X^2] - E[aX]^2 = a^2 E[X^2] - a^2 E[X]^2 = a^2 \text{Var}[X]$.

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- ▶ Uses the same units as X itself.
- ▶ If we switch from feet to inches in our “height of randomly chosen person” example, then X , $E[X]$, and $SD[X]$ each get multiplied by 12, but $\text{Var}[X]$ gets multiplied by 144.

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- ▶ $1 = 1^n = (p + q)^n = \sum_{k=0}^n \binom{n}{k}p^kq^{n-k}$.
- ▶ Number of heads is **binomial random variable with parameters (n, p)** .

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- ▶ Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^n E[X_j] = \sum_{j=1}^n p = np.$$

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- ▶ Can show generally that if X_1, \dots, X_n independent then
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- ▶ We think of a Poisson random variable as being (roughly) a Bernoulli (n, p) random variable with n very large and $p = \lambda/n$.
- ▶ This also suggests $E[X] = np = \lambda$ and $\text{Var}[X] = npq \approx \lambda$.

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- ▶ The numbers of events occurring in disjoint intervals are independent random variables.
- ▶ Probability to see zero events in first t time units is $e^{-\lambda t}$.
- ▶ Let T_k be time elapsed, since the previous event, until the k th event occurs. Then the T_k are independent random variables, each of which is exponential with parameter λ .

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